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All Tabular Generally Post-Complete Extensions of K4 Have Interpolation Property

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Abstract—We prove that if a modal logic L has the interpolation property and contains the formula $p \wedge \Box p \wedge \ldots \Box^n p \to \Box^{n+1} p$, for some $n \in \omega$, then any tabular generally Post-complete extension of L has the interpolation property too.

1. INTRODUCTION

Recall that a modal logic L has *the interpolation property* if $\alpha \to \beta \in L$ implies that there exists γ such that all variables of γ are in both α and β, and $(\alpha \to \gamma) \wedge (\gamma \to \beta) \in L$. So, we can treat the interpolation property as a some “saturation” of L: ‘for any $\alpha \to \beta \in L \ldots$ there exists γ’.

Another notion of “saturation” of modal logics is *generalized Post-completeness*: a logic L is generally Post-complete if L is consistent and has no proper consistent extension preserving admissible inference rules of L.

Below we prove that for tabular extension (nor necessary normal) of **K4** these properties have a link: interpolation property follows from generalized Post-completeness.

We will use only usual notions and other background of modal logic theory which are in [1]. The same reference contains a discussion of notions of generalized Post-completeness and interpolation property, its resent investigations and many others about.

2. THE MAIN RESULT

Though more part of justification of our main statement can be used for not only extensions of **K4**, see the last section, we can’t formulate it here in more general kind, because necessary detail of our proof is

Lemma 1. *The logic* **K4** *has the interpolation property.*

This lemma is the part of the theorem 14.4 [1].

Now let us formulate our main result.

Theorem 1. *Let L is a tabular generally Post-complete extension of* **K4**. *Then L has interpolation property.*

The rest of the section devote a proof the theorem 1.

Recall that for a logic $L \in \mathrm{Ext}\mathbf{K}$, we say that a formula $\alpha(p)$ is *conservative* in Ext$L$ if

$$\alpha(\bot) \wedge \alpha(p) \wedge \alpha(q) \to \alpha(p \to q) \wedge \alpha(\Box p) \in L.$$

In this paper, for brevity, we will suppose that modal formulas construct from propositional variables and the constant $\bot$ with connections $\to$ and $\Box$ only; other usual connections are abbreviations. The principal merit of conservative formulas is that they preserve the interpolation property (also they have other merits), more exactly, we have

Lemma 2. *If L has the interpolation property and formulas α_i, for $i \in I$, are conservative in* Ext$L$, *then $L + \{\alpha_i : i \in I\}$ also has the interpolation property. If L is finitely approximable and formula α is conservative in* Ext$L$, *then $L + \alpha$ also is finitely approximable.*

The Lemma 2 is the item (i) of the theorem 14.5 [1]. The origin of the theorem is [3], where, besides other, have proved the second part of the statement of the Lemma 2 for normal extensions of $\mathbf{K4}$; changes for our case are not essential.

Let us fixed a tabular generally Post-complete logic $L \in \mathrm{Ext}\mathbf{K4}$. By the 2 for us it is enough to axiomatize L under with help conservative formulas only.

At first note that L has the following semantical description.

Lemma 3. *Logic L is a logic of some finite frame $\mathfrak{F} = \langle W, R, D\rangle$[1] where for any point $x \in W$ there is variable-free formula $\chi(x)$ such that:*

—$x \models \chi(x)$;

—if $y \in W$ and $y \neq x$, then $y \not\models \chi(x)$.

Proof. By theorem 13.11 [1] logic L can be defined by a matrix $\langle \mathfrak{A}, \nabla\rangle$ such that $\mathfrak{A}$ is 0-generated modal algebra, ∇ is a proper filter of $\mathfrak{A}$. By theorem 7.75 [1] we can treat that D doesn't included non-trivial normal filter in itself, i.e., if $\nabla' \subseteq \nabla$ is a normal filter, then $\nabla' = \{\top\}$.

Now let us use theorem 7.50 [1] (and its proof) on presentation of finite matrices. Then we obtain that L is defined by a frame $\mathfrak{F} = \langle \mathfrak{A}_+, \nabla_+\rangle$ where the set of points $\mathfrak{A}_+$ consists of atoms of the algebra $\mathfrak{A}$, say the elements $a_1, \ldots, a_n$. The algebra $\mathfrak{A}$ is 0-generated, therefore for all i, $1 \leq i \leq n$ there exist variable-free formulas φ_i that in $\mathfrak{A}$ are true the equalities $\varphi_i = a_i$. Easily to see that now as $\chi(a_i)$ it can to take φ_i.

The Lemma 3 is proved.

So, we can assume that L is defined by a frame $\mathfrak{F} = \langle W, R, D\rangle$ from Lemma 3. Consider the logic L':

$$\begin{aligned} L' = \mathbf{K4} &\oplus \bigvee_{x \in W} \chi(x) \\ &\oplus \bigwedge_{x \in W} (\chi(x) \to \bigwedge_{\substack{y \in W \\ xRy}} \Diamond\chi(y)) \\ &\oplus \bigwedge_{x \in W} (\chi(x) \to \bigwedge_{\substack{y \in W \\ \neg xRy}} \neg\Diamond\chi(y)). \end{aligned}$$

Lemma 4. *For the logic L' are true:*

(i) *$L' \subseteq L$;*
(ii) *L' has the interpolation property;*
(iii) *L' is Kripke-complete (and even is finitely approximable).*

Proof. The item (i) is obvious, because L' is obtained by adding to $\mathbf{K4}$ formulas which are true in $\mathfrak{F}$. Besides, all added formulas are constant ones, therefore by Lemma 1, Lemma 2 and theorem 6.13 [1] we have (ii) and (iii).

The Lemma 4 is proved.

Now for every point $x \in W$ we put

$$u_x(p) = \neg(\Diamond^+(\chi(x) \wedge p) \wedge \Diamond^+(\chi(x) \wedge \neg p)),$$

[1] Recall that for modal logics which are not necessary normal, D from a frame $\mathfrak{F} = \langle W, R, D\rangle$ is a set of designated points; for normal logics already we can suppose $D = W$.

where $\Diamond^+\varphi = \varphi \wedge \Diamond\varphi$.

Lemma 5. *The formula* $\bigwedge_{x\in W} u_x(p)$ *is conservative in* $\mathrm{Ext}L'$.

Proof. Let's to show that

$$\bigwedge_{x\in W} u_x(\bot) \wedge \bigwedge_{x\in W} u_x(p) \wedge \bigwedge_{x\in W} u_x(q) \to \bigwedge_{x\in W} u_x(p \to q) \wedge \bigwedge_{x\in W} u_x(\Box p) \in L'$$

with help of Lemma 4 (iii).

Let's suppose contradiction, i.e., there exists a point d of some frame $\mathfrak{G} = \langle V, Q\rangle$ of logic L' such that

$$d \models \bigwedge_{x\in W} u_x(\bot) \wedge \bigwedge_{x\in W} u_x(p) \wedge \bigwedge_{x\in W} u_x(q) \tag{1}$$

and

$$d \not\models \bigwedge_{x\in W} u_x(p \to q) \tag{2}$$

or

$$d \not\models \bigwedge_{x\in W} u_x(\Box p). \tag{3}$$

Suppose that we have (2). Then for some points $a, b \in V$, $y \in W$ such that $d\overline{Q}a$[2] and $d\overline{Q}b$

$$a \models \chi(y) \wedge (p \to q), \tag{4}$$

$$b \models \chi(y) \wedge \neg(p \to q). \tag{5}$$

From (5) we obtain

$$b \models \chi(y) \wedge p \tag{6}$$

and

$$b \models \chi(y) \wedge \neg q. \tag{7}$$

By (4) one of the following conditions is true:

$$a \models \chi(y) \wedge \neg p \tag{8}$$

or

$$a \models \chi(y) \wedge q. \tag{9}$$

However (8) and (6) give $d \not\models u_y(p)$ and (9) and (7) give $d \not\models u_y(q)$. Both conclusions are contradict (1).

Now suppose (3) is true. Then for some points $a, b \in V$ such that $d\overline{Q}a$ and $d\overline{Q}b$, and $y \in W$ we have

$$a \models \chi(y) \wedge \Box p \tag{10}$$

[2] Recall that $d\overline{Q}a$ means $d = a$ or dQa.

and

$$b \models \chi(y) \wedge \neg\Box q. \tag{11}$$

By (11), there is a point $c \in V$ such that bQc and $c \models \neg p$. Because the formulas added to **K4** for obtaining L' must be true in all points of $\mathfrak{G}$, we have $c \models \chi(z)$, for some $z \in W$ such that yRz. By (10) we have $a \models \chi(y)$ and therefore, by axioms L', $a \models \Diamond\chi(z)$, i.e., there exists a point c' that aQc' and $c' \models \chi(z)$, besides, by (10), $c' \models p$. Finally, by transitivity of Q, we obtain $d \models \Diamond(\chi(z) \wedge p) \wedge \Diamond(\chi(z) \wedge \neg p)$ or, in other symbols, $d \not\models u_z(p)$, that it contradiction with (1).

The Lemma 5 is proved.

Now we require the following logic L'':

$$L'' = L' + \bigwedge_{x \in W} u_x(p).$$

Lemma 6. *For the logic L'' are true:*

(i) *$L'' \subseteq L$;*
(ii) *L'' has the interpolation property;*
(iii) *L'' is Kripke-complete (and even is tabular).*

Proof. The item (i) follows from (i) of Lemma 4 and by that the formula $\bigwedge_{x\in W} u_x(p)$ is true in the frame $\mathfrak{F}$. The item (ii) follows from (ii) of Lemma 4, Lemma 5 and Lemma 2. The item (iii) follows from (iii) of Lemma 4 by Lemma 2. Tabularity of L'' is obtain from that L'' contains formulas $\bigvee_{x\in W} \chi(x)$ ("in every point of a frame one of the formulas $\chi(x)$, $x \in W$, is true") and $\bigwedge_{x\in W} u_x(p)$ ("in every one-generated frame there is at most one point in which the formula $\chi(x)$, $x \in W$, is true").

The Lemma 6 is proved.

Note. The last condition of Lemma 6 can be proved without using of Lemma 5, but by a direct consideration of the canonical models of the logic L''.

Finally, we define the logic L^*:

$$L^* = L'' + \bigvee_{x \in D} \chi(x).$$

Lemma 7. *For the logic L^* are true:*

(i) *L^* has the interpolation property;*
(ii) *$L^* = L$.*

Proof. Obviously, the item (i) follows from (ii) of Lemma 6 and absence of variables in the formula $\bigvee_{x\in D} \chi(x)$.

For the item (ii) we note, at first, that $L^* \subseteq L$ by Lemma 6 and $\bigvee_{x\in D} \chi(x)$ holds in $\mathfrak{F}$. If $L^* \not\vdash \varphi$, then by (iii) of Lemma 6 this formula is refuted in some point d^* of a finite frame of the logic L^*. With help of axioms of L^* easily one can to prove that subframe generated by the point d^* is isomorphic to some subframe of $\mathfrak{F}$ generated by a point $d \in D$, where the isomorphism is a relation associating points in which same formulas $\chi(x)$, $x \in W$, hold. Therefore, $\mathfrak{F} \not\models \varphi$, i.e., $L \not\vdash \varphi$, that gives desired equality.

The Lemma 7 is proved.

By arbitrariness of choice of L the proof of the Theorem 1 is complete.

3. POSSIBLE GENERALIZATIONS, OPEN QUESTIONS

One can to note the following deficiencies of the proved Theorem 1:

—Why the extensions of **K4** are considered, but not extensions of **K**?

—Why the statement says about tabular logics only?

Note that initially the author interests[3] possibility of a proving the theorem for Post-complete logics only, but not generally Post-complete ones.

The tabularity condition in the Theorem 1 is impossible to remove, because the logic

$$\mathbf{GLLin} = \mathbf{K4} \oplus \Box(\Box p \to p) \to \Box p \oplus \Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$$

is generally Post-complete, but has not the interpolation property: by [4] any normal finite width extension of **K4** (**GLLin** has width 1), but of infinite slice, has no the interpolation property. However the author know nothing about similar counterexample among Post-complete logics:

oq1 *Are the Post-complete modal logics (in particular, in extensions of* **K4***) without interpolation property?*

The following question says about one of pretender to a positive answer for the question **oq1**:

oq2 *If* $\mathbf{GLLin} + \Box p \to p$ *has interpolation property?*

Stress that here we take the logic $\mathbf{GLLin} + \Box p \to p$ only on account of proximity to $\mathbf{GLLin}$.

Let's analyze the proof of theorem 1. Where we used **K4**-axiom $\Box p \to \Box\Box p$ and its relational counterpart—transitivity? Besides Lemma 1, such the place is only one: the end of the proof of Lemma 5. Easily to see that in this place the request of transitivity can be weakened by using the formula of the kind

$$p \wedge \Box p \wedge \ldots \Box^n p \to \Box^{n+1} p \tag{12}$$

and corresponding relational counterpart —

$$xR^{n+1}y \to x = y \vee xRy \vee \cdots \vee xR^n y,$$

and changing of the definition of formulas $u_x(p)$:

$$u_x(p) = \neg(\Diamond^{\leq n}(\chi(x) \wedge p) \wedge \Diamond^{\leq n}(\chi(x) \wedge \neg p)),$$

where

$$\Diamond^{\leq n}\varphi = \varphi \vee \Diamond\varphi \vee \cdots \vee \Diamond^n\varphi.$$

So we obtain the following generalization of Theorem 1: *if a modal logic L has the interpolation property and contains at least one formula (12), then any tabular generally Post-complete extension of L has the interpolation property too*. Note that every tabular logic contains a formula (12), therefore we will give a positive answer to the question:

oq3 *If any logic*

$$\mathbf{K} \oplus p \wedge \Box p \wedge \ldots \Box^n p \to \Box^{n+1} p$$

has the interpolation property?

then we will have a positive answer to the question:

oq4 *Is it true that every tabular generally Post-complete modal logic has the interpolation property?*

[3] From the time of preparation of [2].

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