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# Stochastic Models for Communication Networks with Moving Customers

**D.Baum*** **and V. Kalashnikov****

**University of Trier, Dept. IV, D-54286 Trier, Germany,e-mail: baum@uni-trier.de*
***Institute for Information Transmission Problems, Russian Academy of Sciences, 101447 Moscow, Russia, e-mail: vkalash@iitp.ru*

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**Abstract—**Models of integrated services data networks with moving customers (e.g., mobiles with calls in progress) and level dependent spatial Markov-additive processes of arrivals are considered. It is assumed that waiting is not allowed and, accordingly, two types of service models are examined: with infinite number of servers and loss models with finite number of servers. In general, arrivals can be of different types. In all models, service times have arbitrary distribution functions. Basic characteristics of interest are marginal and joint distributions of numbers of customers being served in disjoint spatial subsets. For this characteristics, we derive differential equations and, in some particular cases, obtain their explicit expressions.

**Keywords:** Mobile communication, spatial Markov-additive arrival process, moving customers.

## 1. INTRODUCTION

The modelling of wireless (mobile) communication networks has been restricted so far to the case where customers represent occurrences with invariant local positions. One exception (among few) is the work of Baccelli, Klein, Lebourges, and Zuyev [1, 2] using homogeneous spatial Poisson (point and line) processes and marked Poisson processes in order to reproduce not only static characteristics (the stationary network and the road system) but also dynamics of customers—i.e., the *variation* of the distribution of points (mobile users) over time after their occurrence (traffic model). The selection of initial customer locations, however, takes place in form of a "static" sampling from a point process. It would be more helpful, no doubt, to speak of "random fields" instead of stochastic processes in this respect, since temporal characteristics are included afterwards only via the marking of random fields using velocity distribution parameters.

An approach to including spatial aspects was suggested by Baum and Kalashnikov with respect to batch Markovian arrival processes (BMAPs), see [3], and [4]. As has been shown in [5], this method easily applies to a wider class of processes, in particular to multivariate Markov additive processes of arrivals (MAPAs). Moreover, this approach can successfully be employed to construct queueing models for various dynamic systems with spatial characteristics, including communication networks such as cell networks based on code division multiple access (CDMA) [5, 6]. The movement of customers, however (e.g., mobiles with calls in progress), has not been considered yet.

In this paper, spatial arrival processes over some subset $\mathfrak{R}$ of a complete separable metric space (Polish space) are taken as a basis for producing point patterns over time, and they are combined with the definition of a set of group operations mapping $\mathfrak{R}$ into itself. The modelling technique described in [5,6] can be applied here with minor changes. Further, the transient as well as the equilibrium distributions of the number of moving customers in any Borel subset of $\mathfrak{R}$ (or corresponding equations for them) can be obtained. The main difference to the approach of Baccelli et alii [1,2] lies in the fact that arrivals and departures of customers are modelled by stochastic processes over time (not random fields in a space), which not only reflect lifetime characteristics, but also include the description of spatial distribution as well as dynamical behaviour of customers. Starting point for our research are the results obtained for stations with infinite and finite number of servers in [5] and [6]. These stations are adequate models for CDMA based wireless networks. Any user's request in such networks receives service immediately after its occurrence, at least as long as total capacity tolerates it.

The paper is organized as follows. In Section 2 we introduce basic notation, define spatial arrival processes under consideration, and summarize some properties of these processes. We are interested in the situation when all customers emerge and are served in a subset $\mathfrak{R}$ of the plane $\mathbb{R}^2$. Section 3 is devoted to service systems fed by spatial arrival processes. In Subsection 3.1, a survey of former results for an infinitely many servers system is given. It provides a basis for further investigation of systems with moving customers. Subsection 3.2 refines some results obtained in [5] for loss systems with finite number of servers. In both Subsections 3.1 and 3.2 differential equations for basic characteristics of interest are derived. In Section 4 we consider models with moving customers assuming that the trajectories are constructed by a group mapping of $\mathfrak{R}$ into itself. The movement assumptions are described in Subsection 4.1. In Subsection 4.2 we examine an infinite server model with spatial Cox arrival process and moving customers. Here, it is possible to get explicit expressions for both non-stationary and stationary characteristics (joint distributions of numbers of customers being served in no-overlapping subsets of $\mathfrak{R}$). Subsections 4.3 and 4.4 deal with the $SBMAP/G/\infty$ and the $SMAPA/G/\mathbf{c}/\mathbf{c}$ model with moving customers, respectively. We derive differential equations for the basic system characteristics by generalizing the results of Subsection 3.2. Section 5 contains a short discussion of further generalizations, mentioning also difficulties with respect to practical application of obtained results.

## 2. PRELIMINARIES

### 2.1. Notation

Let us introduce some basic notation used throughout the paper.

Random vectors are denoted by bold face upper case Roman letters ($\mathbf{X}, \mathbf{Y}$, etc.), elements of $\mathbb{R}^d$ are denoted by bold face lower case Roman letters (e.g., $\mathbf{x} = (x_1, x_2, \ldots, x_d)$), and matrices by upper case Roman letters ($A, B$, etc.). Relations $\leq, \geq, <, >$, etc. on $\mathbb{R}^d$ are to be understood to hold for each component. For sequences $\mathfrak{A} = \{A_0, A_1, \ldots\}$, $\mathfrak{B} = \{B_0, B_1, \ldots\}$ of $(m \times m)$-matrices we define a discrete convolution by

$$(\mathfrak{A} * \mathfrak{B})_v = \sum_{\ell=0}^{v} A_\ell B_{v-\ell}.$$

The unit element in the semi-group of such sequences of $(m \times m)$-matrices with respect to the operator "$*$" is the sequence $\mathbf{1} = \{\mathbf{I}, \mathbf{O}, \mathbf{O}, \ldots\}$, where $\mathbf{I}$ and $\mathbf{O}$ are the unit and the null matrices, respectively. The convolutional variant of the exponential function of $\mathfrak{A}t = \{A_0 t, A_1 t, \ldots\}$ (where $t$ is a scalar) is given by

$$e^{*\mathfrak{A}t} = \sum_{\nu=0}^{\infty} (t^\nu/\nu!)\mathfrak{A}^*_\nu\,.$$

### 2.2. Arrival Process

In mobile networks, arrivals can be of different types (e.g., voice, video, etc.) with different arrival laws. In order to describe arrival process in time, one should take into account at least two features: the arrival regime (e.g., its intensity) and characteristics of arriving customers (their service times, type, etc.). This can conveniently be done with the help of a time-homogeneous Markov-additive process of arrivals (MaP). Such a process was considered in Çinlar [7], Prabhu [8], and Pacheco and Prabhu [9]. The state of the process is a pair $(\mathbf{X}, J) = \{(\mathbf{X}_t, J_t) : t \in [0, \infty)\}$, where the *phase component* $J$ is a continuous-time Markov process with a finite state space $E = \{1, \ldots, m\}$. The *additive component* $\mathbf{X}$ takes values in $\mathbb{R}^d$. The MaP $(\mathbf{X}, J)$ is also a continuous-time Markov process with the state space $\mathbb{R}^d \times E$.

A *multivariate Markov additive process of arrivals* (MAPA) is a MaP with the additive component taking values only in the set of nonnegative $d$-dimensional *integer* vectors. A univariate MAPA ($d = 1$) is an ordinary BMAP as introduced by Neuts [10, 11, 12] and Lucantoni [13]. For $d > 1$ we speak of a *multivariate* MAPA. We assume that $\mathbf{X}_0 = \mathbf{0} = (0, \ldots, 0)$ a.s.

Typically, the phase component $J$ determines the arrival characteristic of the current type whereas the additive component $\mathbf{X}$ counts arrivals and may in case contain necessary information about the service process (e.g., if batches in a no-waiting system are interpreted as required amounts of service [6]).

Any MAPA can be defined by the matrix $P_{\mathbf{n}}(t) = (P_{\mathbf{n};ij}(t))_{i,j\in\{1,\ldots,m\}}$ of the transition probabilities

$$P_{\mathbf{n};ij}(t) = \mathbb{P}\left(\mathbf{X}_t = \mathbf{n}, J_t = j \mid \mathbf{X}_0 = \mathbf{0}, J_0 = i\right) .$$

Let $D_{\mathbf{n};ij}$ be the transition intensities from $(\mathbf{k}, i)$ to $(\mathbf{k} + \mathbf{n}, j)$, $i, j \in E$, $\mathbf{k}, \mathbf{n} \in \mathbb{N}_0^d$, $(\mathbf{n}, j) \neq (\mathbf{0}, i)$ and denote by $D_{\mathbf{n}} = (D_{\mathbf{n};ij})_{i,j\in\{1,\ldots,m\}}$ the corresponding intensity matrices. We put

$$\lambda_i = \sum_{(j,\mathbf{n})\neq(i,\mathbf{0})} D_{\mathbf{n};ij}$$

and

$$D_{\mathbf{0};ii} = -\lambda_i \, .$$

We assume, throughout this paper, that the corresponding MAPA is *stable* that is, $\lambda_i < \infty$ for all $i \in E$.

Let us set

$$p_i(\mathbf{n}, j) = \frac{D_{\mathbf{n};ij}}{\lambda_i}, \quad (j, \mathbf{n}) \neq (i, \mathbf{0})$$

and

$$p_i(\mathbf{0}, i) = -1, \quad i \in E \, .$$

Then any MAPA can be treated as follows. The phase component $J_t$ stays at a state $i$ an exponentially distributed time having the parameter

$$\Lambda_i = \lambda_i q_i \, ,$$

where

$$q_i = \sum_{\mathbf{n}, j\colon j\neq i} p_i(\mathbf{n}, j).$$

After this, it jumps to a state $j \neq i$ and a batch of customers $\mathbf{n}$ (may be, $\mathbf{n} = \mathbf{0}$) arrives with the probability $p_i(\mathbf{n}, j)/q_i$. While the phase process $J_t$ stays at a state $i$, the arrival process can be viewed as a superposition of conditionally independent Poisson processes with parameters $\lambda_i p_i(\mathbf{n}, i)$, $\mathbf{n} \neq \mathbf{0}$, the $\mathbf{n}$th process corresponding to arrivals of $\mathbf{n}$-batches. Considering $\mathbf{n}$-batches, one can take into account different types and group arrivals.

If $p_i(\mathbf{n}, j) = 0$ for $j \neq i$ and $\mathbf{n} \neq \mathbf{0}$ then the arrival process is called *Markov modulated*. In it, there is no arrival when the phase component $J$ changes its value. This particular case is of special importance and we will consider it separately in the examples. Let us consider a special univariate case of a Markov modulated process in order to introduce the corresponding notation used in the examples. In this case, $d = 1$. Assume that $p_i(n, j) = 0$ for any $n \geq 2$ that is, the batch size can be only 1. Denote $p_{ij} = p_i(1, j)/q_i$ for $j \neq i$. Then the phase process $J$ can be completely defined by the two collections $(\Lambda_i)_{i\in E}$ and $(p_{ij})_{i,j\in E}$ ($p_{ii} = 0$), where the parameter $\Lambda_i$ defines the time being in state $i$ and $p_{ij}$ is the probability to jump to state $j$ from state $i$ of the process $J$. In order to completely define the Markov modulated process, parameters $\mu_i = \lambda_i p_i(1, i)$ ought to be given. Evidently $\mu_i$ is the conditional arrival intensity provided that the phase is at state $i$. As we have mentioned, for the Markov modulated process, the phase process $J_t$ is itself Markov. If we make no assumption about $J$, but assume that the arrival process is conditionally non-homogeneous Poisson with the intensity $\mu_{J_t}$ provided that a trajectory of $J$ is given, then the arrival process is called *double-stochastic* or *Cox process*. When considering a Cox process, we will not assume necessarily that the state space $E$ of the phase process $J$ is finite.

In the sequel, we have to order elements with vector indices. For this, we define, for any dimension $d$, a bijection $g: \mathbb{N}_0^d \to \mathbb{N}_0$ (see [5, 6]) by

$$g(\mathbf{n}) = \sum_{u=0}^{d-1} \sum_{v=0}^{A_u-1} \binom{d-u+v-1}{d-u-1} \text{ for } \mathbf{n} \neq \mathbf{0}, \quad g(\mathbf{0}) = 0,$$

where $A_u = \sum_{\nu=1}^{d} n_\nu - \sum_{\nu=1}^{u} n_\nu$ for $1 \le u \le d$, $A_0 = \sum_{\nu=1}^{d} n_\nu$. For example, if $d = 2$, then vectors $\mathbf{n} = (n_1, n_2)$ are ordered according to the sequence (0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (3,0), (2,1), (1,2), (0,3), $\ldots$ .

Let $g^{-1}$ be the inverse of $g$, and assume that

$$\Delta = \{D_{g^{-1}(0)}, D_{g^{-1}(1)}, \ldots\},$$
$$\Pi(t) = \{P_{g^{-1}(0)}(t), P_{g^{-1}(1)}(t), \ldots\}$$

are sequences of the corresponding $(m \times m)$-matrices. Then the Chapman–Kolmogorov equations and the Kolmogorov differential equations for a stable MAPA can be written in the following matrix forms as

$$\Pi(s+t) = \Pi(s) * \Pi(t),$$
$$\frac{d}{dt}\Pi(t) = \Delta * \Pi(t)$$

and the solution to the differential equation as $\Pi(t) = e^{*\Delta t}$ (cf. [3, 4]).

For a univariate MAPA, i.e., a BMAP, we do not have to order matrices $D$ and $P$, and similar expressions for transition probabilities can formally be obtained from the expressions above by plugging there $g(n) = n$, $n \ge 0$.

### *2.3. Spatial Arrival Processes*

The following construction of spatial Markov-additive processes of arrivals (SMAPA) is similar to the particular case of spatial BMAP (SBMAP) which, in its most general form, was constructed by Breuer [14], who admitted the phase space to be uncountable (continuous). This version, however, is beyond of our scope. For a finite phase space $E$ a definition of SBMAP was given by Baum and Kalashnikov [4] (cf. also [3] for a rudimentary version). Following to these constructions, a rough characterization of an SMAPA can be viewed as a generalization of a MAPA, whose rate matrices are equipped with probability measures over Borel subsets of a region $\mathfrak{R}$ of some Polish space $\mathbb{S}$. We will assume, for definiteness, that $\mathbb{S} = \mathbb{R}^2$.

Consider a MAPA defined by the sequence $\Delta$. Let

$$\Phi = \{\phi_{\mathbf{n};ij} : i, j \in E, \mathbf{n} \in \mathbb{N}_0^d\}$$

be a family of probability measures over $\mathcal{B}(\mathfrak{R})$, the $\sigma$-algebra of Borel subsets of $\mathfrak{R}$. Then the spatial MAPA (SMAPA) is defined by its rate matrices for any $S \in \mathcal{B}(\mathfrak{R})$:

$$\begin{aligned} D_{\mathbf{n};ij}(S) &= \lambda_i p_i(\mathbf{n}, j) \phi_{\mathbf{n};ij}(S), \quad \mathbf{n} \neq \mathbf{0}, \\ D_{\mathbf{0};ij}(S) &= \lambda_i p_i(\mathbf{0}, j) + \lambda_i \sum_{\mathbf{n} \neq \mathbf{0}} p_i(\mathbf{n}, j) \phi_{\mathbf{n};ij}(\mathfrak{R} \setminus S) \end{aligned} \tag{2.1}$$

(notice that we formally set $p_i(0, i) = -1$ for $i \in E$). Loosely speaking, the spatial generalization of a MAPA consists of the additional assumption that each nonempty $\mathbf{n}$-batch arriving at a time when the phase process of the MAPA jumps from state $i$ to state $j$ (may be, $i = j$) is nested at a subset $S$ with the probability $\phi_{\mathbf{n};ij}(S)$.

Let $\mathbf{X}_t(S)$ be the total number of vector-valued arrivals at a subset $S$ during time $t$. Then $(\mathbf{X}_t(S), J_t)$ is a homogeneous Markov process for each Borel $S$. Let

$$P_{\mathbf{n};ij}(S,t) = \mathbb{P}\{\mathbf{X}_t(S) = \mathbf{n}, J_t = j \mid \mathbf{X}_0(S) = \mathbf{0}, J_0 = i\}$$

be its transition probability. Similarly to the Subsection 2.2, the sequence

$$\Pi(S,t) = \{P_{g^{-1}(0)}(S,t), P_{g^{-1}(1)}(S,t), P_{g^{-1}(2)}(S,t), \dots\}$$

is again given by

$$\Pi(S,t) = e^{*\Delta(S)t}$$

where

$$\Delta(S) = \{D_{g^{-1}(0)}(S), D_{g^{-1}(1)}(S), D_{g^{-1}(2)}(S), \dots\}.$$

Let now $\mathbf{S} = \{S_1, S_2, \dots, S_\kappa\}$, $\kappa > 0$, be a family of non-overlapping Borel subsets of $\mathfrak{R}$ and set

$$\vec{\mathbf{X}}_t(\mathbf{S}) = \{\mathbf{X}_t(S_1), \mathbf{X}_t(S_2), \dots, \mathbf{X}_t(S_\kappa)\}.$$

Furthermore, let

$$P_{\vec{\mathbf{n}};ij}(\mathbf{S},t) = \mathbb{P}\left(\vec{\mathbf{X}}_t(\mathbf{S}) = \vec{\mathbf{n}}, J_t = j \mid \vec{\mathbf{X}}_0(\mathbf{S}) = \mathbf{0}, J_0 = i\right), \tag{2.2}$$

where $\vec{\mathbf{n}} = (\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_\kappa)$ and each $\mathbf{n}_c$ ($1 \le c \le \kappa$) is an $d$-dimensional integer vector. Assume that $P_{\vec{\mathbf{n}}}(\mathbf{S},t)$ is a matrix with elements (2.2). Consider sequences

$$\Pi(\mathbf{S},t) = \{P_{\vec{\mathbf{n}}}(\mathbf{S},t)\}_{\vec{\mathbf{n}} \ge \mathbf{0}}$$

and

$$\Delta(\mathbf{S}) = \{D_{\vec{\mathbf{n}}}(\mathbf{S})\}_{\vec{\mathbf{n}} \ge \mathbf{0}}.$$

Define for them the convolution operation as follows

$$\left(\Delta^{*0}(\mathbf{S})\right)_{\vec{\mathbf{n}}} = \begin{cases} \mathbf{I}, & \text{if} \quad \vec{\mathbf{n}} = \mathbf{0}, \\ \mathbf{O}, & \text{otherwise,} \end{cases}$$

$$\left(\Delta^{*1}(\mathbf{S})\right)_{\vec{\mathbf{n}}} = (\Delta(\mathbf{S}))_{\vec{\mathbf{n}}} \quad \text{for all} \quad \vec{\mathbf{n}} \in \mathbb{N}_0{}^{\kappa+d},$$

$$\left(\Delta^{*k}(\mathbf{S})\right)_{\vec{\mathbf{n}}} = \sum_{\ell=0}^{g(\vec{\mathbf{n}})} \left(\Delta^{*k-1}(\mathbf{S})\right)_{\vec{\mathbf{n}}-g^{-1}(\ell)} D_{g^{-1}(\ell)}(\mathbf{S}) \ \text{ for } k \ge 1.$$

The following theorem was proved in [4].

**Theorem 1.** *If $S_1, S_2, \dots, S_K$ are mutually non-overlapping Borel subsets, then the joint distribution of the components of the random vector $\{\vec{\mathbf{X}}_t(\mathbf{S}), J_t\}$ is given in a convolutional exponential form as*

$$\Pi(\mathbf{S},t) = e^{*\Delta(\mathbf{S})t},$$

$$P_{\mathbf{n}}(\mathbf{S},t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \left(\Delta^{*\nu}(\mathbf{S})\right)_{\mathbf{n}}.$$

Let us return to a particular case of the Markov modulated process that was defined in Subsection 2.2 by the collections of parameters $\Lambda_i$, $p_{ij}$, and $\mu_i$. In order to define a spatial version of this process, one can additionally define a collection $(\phi_i(S))_{i\in E}$ of probability measures on Borel subsets of $\mathfrak{R}$. The quantity $\phi_i(S)$ is the probability that a customer who arrived at a time when the phase process is at a state $i \in E$ is placed at a subset $S$. Similarly, a Cox process can be generalized to a spatial Cox process by defining a family $(\phi_i(S))_{i\in E}$ where $E$ is not necessarily finite or even denumerable.

## 3. SERVING SPATIAL ARRIVAL PROCESSES

In mobile networks, waiting for service typically is not possible. Therefore, service systems without waiting rooms are of primary practical interest. We focus on a system with infinitely many servers, where all customers receive immediate service, and on a loss system with finite number of servers. In this section we recapitulate some results from [5] and [6] for non-moving customers.

### *3.1. The $SBMAP/G/\infty$ System*

Consider the $SBMAP/G/\infty$ model, i.e., the system with univariate arrival process and infinitely many servers. Following to [6], let us fix a Borel subset $S \subset \mathfrak{R}$. Denote by $0 < T_0 \le T_1 \le T_2 \le \dots$ successive arrival epochs of customers to this subset $S$. Clearly, $T_i$ can be equal to $T_j$ if and only if both the $i$th and the $j$th customers belong to the *same* batch. We assume that separate customers occupy identical servers, and their service times are iid random variables with a common distribution function $F$, $F(0) = 0$. Because of this, the order of customers inside batches is of no importance. Let $B_i$ be the service time of the $i$th customer. Define indicator functions

$$\chi_i^S(u) = \begin{cases} 1, & \text{if} \quad T_i \le u < T_i + B_i, \\ 0, & \text{otherwise,} \end{cases}$$

for all $i \ge 1$. Then $\chi_i^S(u) = 1$ if and only if the $i$th customer is in service at time $u$. Let, for $u \le t$,

$$N_{u,t}(S) = \sum_{i:\, T_i \le u} \chi_i^S(t)$$

be the number of customers arrived to $S$ until time $u$ and still being in service at time $t$. In particular,

$$N_u(S) = N_{u,u}(S)$$

is the number of customers in the system at time $u$. Assume that $N_0(S) = 0$. Evidently, the process $(N_{u,t}(S), J_u)$, $0 \le u \le t$, is Markov, for any fixed $t$. Let

$$Q_{r;ik}(S;\, u,t) = \mathbb{P}\left(N_{u,t}(S) = r, J_u = k \mid J_0 = i\right)$$

In order to write differential equations for functions $Q_{r;ik}(S;\, u,t)$, we consider their infinitesimal increments over $[u, u+du]$ and take into account the following fact: if, at time $u$, a batch of size $n$ arrives, then

$$b_k(n, F(t-u)) = \binom{n}{k} \left(1 - F(t-u)\right)^k F^{n-k}(t-u), \quad 0 \le k \le n, \tag{3.1}$$

is the probability that exactly $k$ customers from these $n$ will be resident at time $t$.

Let $Q_r(S;\, u,t)$ stand for the matrix with elements $Q_{r;ik}(S;\, u,t)$, and assume that

$$\mathcal{Q}(S;\, u,t) = \{Q_0(S;\, u,t), Q_1(S;\, u,t), \dots\}$$

is the corresponding sequence of matrices. Denote

$$\begin{aligned} Q_r(S;\, t) &= Q_r(S;\, t,t)\,, \\ \mathcal{Q}(S;\, t) &= \mathcal{Q}(S;\, t,t)\,, \\ R_k(S;\, v) &= \sum_{n=k}^{\infty} D_n(S) b_k(n, F(v)), \quad v \ge 0,\ k \ge 0, \\ \mathcal{R}(S;\, v) &= \{R_0(S;\, v), R_1(S;\, v), \dots\}\,. \end{aligned}$$

Then the following matrix differential equation holds:

$$\frac{\partial}{\partial u} Q_r(S;\, u, t) = \left(\mathcal{Q}(S;\, u, t) * \mathcal{R}(S;\, t-u)\right)_r . \tag{3.2}$$

According to Theorem 1 in [6], the sequence of transient state probabilities $\mathcal{Q}(S;\, u, t)$ can be computed by iterations:

$$\mathcal{Q}^{(0)}(S;\, u, t) = \mathbf{1},$$
$$\mathcal{Q}^{(i+1)}(S;\, u, t) = \mathbf{1} + \int_0^u \mathcal{Q}^{(i)}(S;\, v) * \mathcal{R}(S;\, t-v)\, dv\,, \quad i \geq 0\,,$$

where $\mathbf{1} = \{\mathbf{I}, \mathbf{O}, \mathbf{O}, \ldots\}$. It is well known that these iterations converge to a unique sequence

$$\mathcal{Q}(S;\, u, t) = \lim_{i\to\infty} \mathcal{Q}^{(i)}(S;\, u, t)\,. \tag{3.3}$$

(see, e.g., [15] for a matrix analog of equation (3.1)).

The equilibrium distribution exists for each stable BMAP and each service time distribution $F$ with the finite mean. It is given by the probabilities

$$Q_{r;j}(S) = \lim_{t\to\infty} Q_{r;ij}(S;\, t), \quad \forall\, j \in E, r \geq 0\,, \tag{3.4}$$

independent of the initial phase $J_0 = i$.

These results can substantially be simplified in the Markov modulated case. We will consider this case separately in Section 4.2, where explicit solutions are obtained and further generalizations are indicated.

### *3.2. The $SMAPA/G/\mathbf{c}/\mathbf{c}$ System*

In this subsection we assume that the arrival process is SMAPA (as it was described in Subsection 2.3) defined by matrices $D_{\mathbf{n}}(S)$. In case of a multivariate SMAPA, arrivals transport batches of vectors $\mathbf{n} = (n_1, \ldots, n_d)$ into the system. The components $n_i,\ i \in \{1, \ldots, d\}$, can be interpreted as customer class specific batch arrivals. There are $c_i$ servers available for each class $i \in \{1, \ldots, d\}$, and we assume that the vector $\mathbf{c} = (c_1, \ldots, c_d)$ additionally determines a capacity restriction for the whole system, each component describing a class specific restriction. This means that in $\mathfrak{R}$ there cannot be more than $\mathbf{c}$ customers simultaneously, and that all arrivals to the filled system are lost. The actual numbers $N_{t,i}$ of class-$i$ customers in the system at time $t$ define what is called the *level vector* $\mathbf{N}_t = (N_{t,1}, \ldots, N_{t,d})$ at time $t$ (or the *level* for short).

Accordingly, for each $S \in \mathcal{B}(\mathfrak{R})$, we write $\mathbf{N}_t(S) = (N_{t,1}(S), \ldots, N_{t,d}(S))$ for the vector of customer numbers of different classes in subset $S$ at time $t$ (we may say that $N_{t,i}(S)$ is the number of class-$i$ customers being served at subset $S$ at time $t$). The set of possible system levels is

$$\mathcal{N} = \{\mathbf{k} :\ \mathbf{0} \leq \mathbf{k} \leq \mathbf{c}\}\,.$$

If, at some moment, the system level is $\mathbf{k} = (k_1, \ldots, k_d)$ and, at this time, a batch $\mathbf{n}$ of new customers arrived, then the new system level becomes $\mathbf{m} = (m_1, \ldots, m_d)$ with $m_i = \min(k_i + n_i, c_i)$, $1 \leq i \leq d$. Bearing this in mind, let us define, for each $\mathbf{k} \leq \mathbf{c}$ and $\mathbf{n} \leq \mathbf{c} - \mathbf{k}$, a set of batches $B(\mathbf{k}, \mathbf{n})$ as follows. A vector $\mathbf{m} = (m_1, \ldots, m_d)$ belongs to $B(\mathbf{k}, \mathbf{n})$ if and only if its components satisfy $m_c = n_c$ if $n_c < c_c - k_c$ or $m_c \geq n_c$ if $n_c = c_c - k_c$, $1 \leq c \leq d$. Put

$$D_{\mathbf{n}}^{(\mathbf{k})}(\mathfrak{R}) = \sum_{\mathbf{h}\in B(\mathbf{k},\mathbf{n})} D_{\mathbf{h}}(\mathfrak{R})\,, \tag{3.5}$$

where $D_{\mathbf{h};ij}(\mathfrak{R}) = \lambda_i p_i(\mathbf{n}, j)$, the matrices $D_{\mathbf{n}}(S)$ being defined according to (2.1). Suppose that service times for different classes have different distributions in general. More specifically, let $F_c$ be a service time distribution function for customers of class $c$, $1 \le c \le d$, $F_c(0) = 0$. We will also use the renewal functions

$$H_c(u) = \sum_{k=1}^{\infty} F_c^{[k]}(u)$$

and their densities (assuming that they exist)

$$h_c(u) = \frac{dH_c(u)}{du} \quad \text{for} \ \ u \ge 0\,,$$

where $F_c^{[k]}$ is the $k$-fold convolution of $F_c$.

Let, for $\mathbf{r} \le \mathbf{l} \le \mathbf{c}$ and $i, j \in E$, $Q^{(\mathbf{l})}_{\mathbf{r};ij}(S;\, u, t)$ denote the probability that the state (the system level and the phase) is $(\mathbf{l}, j)$ at time $u \le t$, and $\mathbf{r}$ the vector of class specific customers in $S$ remaining resident in $S$ up to time $t$, given that the process starts from a state with zero customers in $\mathfrak{R}$ and the phase being $i$. We write $Q^{(\mathbf{l})}_{\mathbf{r}}(S;\, u, t)$ for the corresponding matrix, i.e.,

$$Q^{(\mathbf{l})}_{\mathbf{r}}(S;\, u, t) = \Big( (Q^{(\mathbf{l})}_{\mathbf{r};ij}(S;\, u, t)) \Big)_{i,j \in E}\,.$$

For a *non-spatial* arrival process (one can formally regard that, in this case, $S = \mathfrak{R}$ and therefore $S$ can be omitted), the following theorem was proven in [5].

**Theorem 2.** *Let* $0 \le u \le t$ *and* $\mathbf{0} \le \mathbf{r} \le \mathbf{l} \le \mathbf{c}$. *Then the matrix* $Q^{(\mathbf{l})}_{\mathbf{r}}(u, t)$ *satisfies the differential equation*

$$\frac{\partial Q^{(\mathbf{l})}_{\mathbf{r}}(u,t)}{\partial u} = \sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{r}} \sum_{\mathbf{k}=\mathbf{m}}^{\mathbf{l}-(\mathbf{r}-\mathbf{m})} Q^{(\mathbf{k})}_{\mathbf{m}}(u,t) D^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k}} \prod_{c=1}^{d} b_{r_c - m_c}(l_c - k_c; F_c(t-u))$$
$$+ \sum_{c=1}^{d} h_c(u) \left( Q^{(\mathbf{l}+\mathbf{e}_c)}_{\mathbf{r}}(u,t)(l_c + 1 - r_c)\delta_{\mathbf{c}-\mathbf{l}-\mathbf{e}_c} - \ Q^{(\mathbf{l})}_{\mathbf{r}}(u,t)(l_c - r_c) \right)\,,$$

*where the functions* $b_{r_c - m_c}$ *are defined in (3.1),* $\mathbf{e}_c$ *is an* $d$*-dimensional vector consisting of zeros except the cth component which is equal to 1, and* $\delta_{\mathbf{x}}$ *is the Kronecker function, i.e.,* $\delta_{\mathbf{x}} = 1$, *if* $\mathbf{x} \ge \mathbf{0}$, *and zero otherwise.*

In order to derive similar differential equations for the case of a spatial arrival process we consider a time interval $[u, u + du]$ and take into account that, during this interval, at most one arrival can occur (if one neglect terms $o(du)$). An arrival in $S$ can change the number of $t$-resident customers as well as the system level, but an arrival in $\mathfrak{R} \setminus S$ may change only the system level. As a consequence, as long as the vector $\mathbf{m}$ of $t$-resident customers in $S$ at time epoch $u$ is less than the corresponding vector $\mathbf{r}$ at time epoch $u + du$, only arrivals to subset $S$ have to be considered, whereas for $\mathbf{m} = \mathbf{r}$ also arrivals to the complement of $S$ have to be counted.

Let

$$\begin{aligned} U^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k},\mathbf{r}-\mathbf{m}}(S\,; u, t) &= D^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k}}(S) \prod_{c=1}^{d} b_{r_c - m_c}(l_c - k_c; F_c(t-u))\,, \\ V^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k}}(S;\, u, t) &= D^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k}}(S) \prod_{c=1}^{d} b_0(l_c - k_c; F_c(t-u)) + D^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k}}(\mathfrak{R} \setminus S)\,, \\ W^{(\mathbf{l},\mathbf{e}_c)}_{c;\,\mathbf{r}}(S\,; u, t) &= Q^{(\mathbf{l}+\mathbf{e}_c)}_{\mathbf{r}}(S;\, u, t)(l_c + 1 - r_c)\delta_{\mathbf{c}-\mathbf{l}-\mathbf{e}_c} \\ &\qquad - Q^{(\mathbf{l})}_{\mathbf{r}}(S;\, u, t)(l_c - r_c)\,. \end{aligned} \tag{3.6}$$

The corresponding differential equation then becomes

$$\frac{\partial Q_{\mathbf{r}}^{(\mathbf{l})}(S;\,u,t)}{\partial u} = \sum_{\substack{\mathbf{0}\le\mathbf{m}\le\mathbf{r}\\ \mathbf{m}\neq\mathbf{r}}} \sum_{\mathbf{k}=\mathbf{m}}^{\mathbf{l}-(\mathbf{r}-\mathbf{m})} Q_{\mathbf{m}}^{(\mathbf{k})}(S;\,u,t)U_{\mathbf{l}-\mathbf{k},\,\mathbf{r}-\mathbf{m}}^{(\mathbf{k})}(S\,;u,t)$$

$$+\, Q_{\mathbf{r}}^{(\mathbf{l})}(\mathfrak{R};\,u,t)D_{\mathbf{0}}^{(\mathbf{l})}(S) + \sum_{\substack{\mathbf{r}\le\mathbf{k}\le\mathbf{l}\\ \mathbf{k}\neq\mathbf{l}}} Q_{\mathbf{r}}^{(\mathbf{k})}(S;\,u,t)V_{\mathbf{l}-\mathbf{k}}^{(\mathbf{k})}(S;\,u,t)$$

$$+\sum_{c=1}^{d} h_c(u)W_{c;\,\mathbf{r}}^{(\mathbf{l},\mathbf{e}_c)}(S\,;u,t)\,. \tag{3.7}$$

Recall that the inequality $\mathbf{m}\le\mathbf{r}$ with $\mathbf{m}\neq\mathbf{r}$ in (3.7) means that each component of $\mathbf{m}$ is less or equal to the corresponding component of $\mathbf{r}$, while at least one component of $\mathbf{r}$ is greater than the corresponding component of $\mathbf{m}$.

The sum

$$Q_{\mathbf{r}}(S;\,u,t) = \sum_{\mathbf{l}\ge\mathbf{r}} Q_{\mathbf{r}}^{(\mathbf{l})}(S;\,u,t)$$

contains complete information about $t$-resident customers observed at epoch $u\le t$ in $S$, and the transient state probability matrices can be obtained as $Q_{\mathbf{r}}(S;t) = Q_{\mathbf{r}}(S;\,t,t)$. Since the model under consideration is stable, these matrices tend to finite limits as $t\to\infty$, providing a sequence of equilibrium state probability vectors with matrix components

$$\mathcal{Q}_{\mathbf{r}}(S) = (Q_{\mathbf{r};1}(S), Q_{\mathbf{r};2}(S),\,\dots\,, Q_{\mathbf{r};m}(S)),$$

where

$$Q_{\mathbf{r};j}(S) = \lim_{t\to\infty} Q_{\mathbf{r};ij}(S;t),\quad i\in E=\{1,\dots,m\},\ \mathbf{0}\le\mathbf{r}\le\mathbf{c}\,.$$

In order to find a solution to the differential equation (3.7), we follow the trick used in [5] where the corresponding equation for the non-spatial case was transformed into a homogeneous matrix-vector differential equation by ordering the doubly indexed (matrix) structures into single-indexed (vector) structures. Let $\beta:\ \mathbb{N}_0^d\times\mathbb{N}_0^d\to\mathbb{N}_0$ denote a function that uniquely maps a pair $(\mathbf{l},\mathbf{r})\in\{(\mathbf{x},\mathbf{y}):\ \mathbf{0}\le\mathbf{x},\mathbf{y}\le\mathbf{c}\}$ of vector indices to an integer $\beta(\mathbf{l},\mathbf{r})$. To be more precise, let $K$ be a total number of possible system levels,

$$K=\prod_{i=1}^{d}(1+c_i),$$

and $\beta$ denote a one-to-one correspondence between the set $\{(\mathbf{x},\mathbf{y}):\ \mathbf{0}\le\mathbf{x},\mathbf{y}\le\mathbf{c}\}$ and the set of integers $\{0,1,2,\dots,a_K\}$ with $a_K=K^2-1$. Obviously, there are several possibilities to define $\beta$ (the function $g$, however, as defined in Section 2 is inadequate here because of the capacity restriction). The next step is to order matrices $Q_{\mathbf{r}}^{(\mathbf{l})}(S;\,u,t)$ by means of $\beta$, denoting

$$Q_{\mathbf{r}}^{(\mathbf{l})}(S;\,u,t) = Q^{[\beta(\mathbf{l},\mathbf{r})]}(S;\,u,t),\quad \mathbf{0}\le\mathbf{r}\le\mathbf{l}\le\mathbf{c}\,,$$

where

$$Q_{\mathbf{m}}^{(\mathbf{n})}(S;\,u,t)=\mathbf{O},\quad \text{if }\ \mathbf{m}\not\le\mathbf{n}\ \text{ or } \mathbf{n}\not\le\mathbf{c}\,,$$

and to define the sequence

$$\mathfrak{Q}^{[\beta]}(S;u,t) = (Q^{[0]}(S;\,u,t), Q^{[1]}(S;\,u,t),\,\dots,\,Q^{[a_K]}(S;\,u,t))\,.$$

Then the following theorem is an immediate consequence from (3.7) (compare also [5], and refer to the above definition of $U_{\mathbf{l}-\mathbf{k},\mathbf{r}-\mathbf{m}}^{(\mathbf{k})}(S;\,u,t)$ and $V_{\mathbf{l}-\mathbf{k}}^{(\mathbf{k})}(S;\,u,t)$).

**Theorem 3.** *For $\mathbf{r} \leq \mathbf{l} \leq \mathbf{c}$ and $u \leq t$, the equation (3.7) takes the form*

$$\frac{\partial \mathfrak{Q}^{[\beta]}(S;\, u, t)}{\partial u} = \mathfrak{Q}^{[\beta]}(S;\, u, t)\mathcal{H}_{\mathbf{c}}(S;\, u, t)\,, \tag{3.8}$$

*where $\mathcal{H}_{\mathbf{c}}(S; u, t)$ is a $(a_K \times a_K)$-matrix of $(m \times m)$-matrices defined as follows. Assume that $\beta^{-1}(i) = (\mathbf{m}, \mathbf{k})$ and $\beta^{-1}(j) = (\mathbf{r}, \mathbf{l})$. Then the $(i, j)$-entry in $\mathcal{H}_{\mathbf{c}}(S;\, u, t)$ (denoted below for simplicity as $(\mathcal{H})_{i,j}$) is represented by the following expression:*

$$(\mathcal{H})_{i,j} = \begin{cases} \mathbf{O}, & if\ \mathbf{m} \not\leq \mathbf{k},\ or\ \mathbf{r} \not\leq \mathbf{l},\ or\ \mathbf{m} \not\leq \mathbf{r}, \\ U^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k},\mathbf{r}-\mathbf{m}}(S;\, u, t), & if\ \mathbf{m} < \mathbf{r}, \mathbf{m} \leq \mathbf{k} \leq \mathbf{l} - (\mathbf{r} - \mathbf{m}), \\ V^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k}}(S;\, u, t), & if\ \mathbf{m} = \mathbf{r}, \mathbf{r} \leq \mathbf{k}, \mathbf{k} \not\geq \mathbf{l}, \\ D^{(\mathbf{l})}_{\mathbf{0}}(\mathfrak{R}) - \mathbf{I}\sum_{c=1}^{d} h_c(u)(l_c - r_c), & if\ \mathbf{m} = \mathbf{r}\ and\ \mathbf{k} = \mathbf{l}, \\ \mathbf{I}\sum_{c=1}^{d} h_c(u)(l_c - r_c + 1), & if\ \mathbf{m} = \mathbf{r}\ and\ \mathbf{k} = \mathbf{l} + \mathbf{e}_c, \\ & and\ \ \mathbf{c} - (\mathbf{l} + \mathbf{e}_c) \geq \mathbf{0}, \\ \mathbf{O}, & in\ all\ other\ cases. \end{cases} \tag{3.9}$$

**Theorem 4.** *The solution to (3.8) has the form*

$$\mathfrak{Q}^{[\beta]}(S;\, u, t) = \mathfrak{Q}^{[\beta]}(S; 0, t)\, \mathcal{X}(S;\, u, t)\,,$$

*where $\mathcal{X}$ is an $(a_K \times a_K)$-block matrix with $\mathcal{X}(S; 0, t) = \mathbf{I}$, satisfying the same differential equation (3.8). It can be found as the limit (as $i \to \infty$) of the following successive approximations:*

$$\begin{aligned} \mathcal{X}_0(S;\, u, t) &= \mathbf{I}, \\ \mathcal{X}_{i+1}(S;\, u, t) &= \mathbf{I} + \int_{s=0}^{u} \mathcal{X}_i(S; s, t)\mathcal{H}_{\mathbf{c}}(S;\, s; t)\, ds \quad i \geq 0\,. \end{aligned} \tag{3.10}$$

Given $\mathfrak{Q}^{[\beta]}(S;\, u, t)$ for $u = t$, one can find the transient state probability matrices with elements

$$Q_{\mathbf{r};i,j}(S; t) = \mathbb{P}\left(\mathbf{N}_t(S) = \mathbf{r}, J_t = j \mid \mathbf{N}_0(S) = \mathbf{0}, J_0 = i\right)\,, \tag{3.11}$$

using the equality

$$Q_{\mathbf{r}}(S; t) = \sum_{\mathbf{l} \geq \mathbf{r}} Q^{[\beta(\mathbf{r},\mathbf{l})]}(S;\, t, t) = \sum_{\mathbf{l} \geq \mathbf{r}} Q^{(\mathbf{l})}_{\mathbf{r}}(S;\, t, t)\,. \tag{3.12}$$

The equilibrium state probabilities can be found as the limits

$$Q_{\mathbf{r};j}(S) = \lim_{t\to\infty} Q_{\mathbf{r};ij}(S; t)\,,$$

that exist due to the stability of the queue.

## 4. SERVING MOVING CUSTOMERS

### *4.1. Modelling of Movements*

Assume that some spatial arrival process is given (SMAPA, SBMAP, spatial Markov modulated, or spatial Cox process). And let each arriving customer immediately, upon arrival, start moving in space in accordance with the law

$$\mathbf{x}(s) = \Upsilon_s(\mathbf{x})\,, \tag{4.1}$$

where $\mathbf{x}$ is its position upon arrival, and $\mathbf{x}(s)$ is its position at time $s$. Actually, values $\mathbf{x}(s)$ are defined for $s \geq 0$ but it is convenient to define them for all $-\infty < s < \infty$. For this, we assume that $\{\Upsilon_s\}_{-\infty<s<\infty}$ is a collection of group operations mapping the space $\mathfrak{R}$ into itself. In particular, $\Upsilon_{s+t} = \Upsilon_s\Upsilon_t$ for all $s, t$. Let

$$\Upsilon_s[S] = \{\mathbf{y} : \mathbf{y} = \Upsilon_s(\mathbf{x}), \mathbf{x} \in S\}, \ s \geq 0,$$

and

$$\Upsilon^{(-s)}[S] = \{\mathbf{x} : \Upsilon_s(\mathbf{x}) \in S\}, \ s \geq 0$$

for each measurable subset $S$ of $\mathfrak{R}$. By the assumption that $\Upsilon_s$ forms a group, we get that $\Upsilon^{(-s)}[S] = \Upsilon_{-s}[S]$ for all $-\infty < s < \infty$. The curves $(\Upsilon_s(\mathbf{x}))_{-\infty<s<\infty}$ ($\mathbf{x} \in \mathfrak{R}$) can be viewed as the traces along which customers are moving.

In applications there may be several streams of moving customers as well as sets of non-moving customers, movements may occur in different directions, and the directions can be chosen randomly. In order to consider these possibilities, one can consider several group operations and introduce a random mechanism of their choosing. For simplicity, we do not consider such possibilities in this paper leaving this for future.

### *4.2. Spatial Cox Arrivals with Infinitely Many Servers*

Let us consider a system consisting of infinitely many servers serving a spatial Cox arrival process defined by a phase process $J$ with the state space $E$ (not necessarily finite), intensities $(\mu_i)_{i\in E}$, and a collection of measures $(\phi_i(S))_{i\in E}$, where $S$ is a Borel subset of $\mathfrak{R}$. We assume that $\{J_t\}_{t\geq 0}$ is a random process with cadlag trajectories (right-continuous and having limits from the left at each point $t$). By the definition of a spatial Cox process, customers appear in time and space in accordance with the intensity $\xi_{J_u}(S) = \mu_{J_u}\phi_{J_u}(S)$, meaning that the probability for a customer to arrive during time interval $[u, u+\Delta u]$ in a subset $S \subset \mathfrak{R}$ is $\xi_{J_u}(S)\Delta u + o(\Delta u)$. Assume further that each customer moves in space in accordance with the law (4.1).

Assume, additionally, that each customer appearing at a time epoch, when the phase process $J$ is in state $j \in E$, is served according to the distribution function $F_j(u)$ independently of other customers. Set $\overline{F}_j = 1 - F_j$. Our goal is to find time-spatial probabilities like

$$Q_{k_1,\dots,k_n}(S_1, \dots, S_n; t) = \mathbb{P}\left(N_t(S_1) = k_1, \dots, N_t(S_n) = k_n\right), \tag{4.2}$$

where $N_t(S_i)$ is the number of customers in subset $S_i$ at time $t$, and all $S_i$ are disjoint. We shall also calculate moments as well as stationary characteristics of the above distribution. Let

$$\xi_t = \xi_{J_t}(\mathfrak{R}) \tag{4.3}$$

and

$$\Xi_t = \int_0^t \xi_u \, du\,. \tag{4.4}$$

**Lemma 1.** *In the case $n = 1$ in (4.2), $N_t(S)$ has the following distribution*

$$Q_k(S;t) = \mathbb{P}\left(N_t(S) = k\right) = \frac{1}{k!}\,\mathbb{E}\left[\Theta^k(S;t)\exp(-\Theta(S;t))\right], \quad t \geq 0\,, \tag{4.5}$$

*where*

$$\Theta(S;t) = \int_0^t \xi_{J_{t-u}}(\Upsilon_{-u}[S])\overline{F}_{J_{t-u}}(u)\, du\,. \tag{4.6}$$

**Proof.** Let us fix $t > 0$ and the trajectory of the modulating process $\{J_u\}_{0 \le u \le t}$. Then the total number of customers $X_t$ that arrived within $[0, t)$ follows the Poisson distribution (conditioned by $\{J_u\}_{0 \le u \le t}$):

$$\mathbb{P}(X_t = n) = \frac{\Xi_t^{\,n}}{n!} \exp(-\Xi_t).$$

Given $X_t$, all customers can be regarded as independent, and their arrival times are distributed in accordance with the density $\xi_u/\Xi_t$, $0 \le u \le t$. Furthermore, if a customer arrives at time $u$, then it occurs in a subset $S$ at time $t$ if and only if

(i) its virtual position at time $t$ belongs to $S$ that is, it emerged in a subset $\Upsilon_{u-t}[S]$;

(ii) its service time is greater than $t - u$, the probability of this is $\overline{F}_{J_u}(t - u)$.

Then $q(S; t) = \Theta(S; t)/\Xi_t$ (see (4.6)) is a conditional probability that a customer will be in a subset $S$ at time $t$. This yields

$$\mathbb{P}\left(N_t(S) = k \mid \{J_u\}_{0 \le u \le t}\right) = \sum_{n \ge k} \frac{1}{n!}\, \Xi_t^{\,n} e^{-\Xi_t} \binom{n}{k} q^k(S; t)(1 - q(S; t))^{n-k},$$

and, consequently, (4.5).

We can rewrite the result of Lemma 1 in terms of generating functions as follows.

**Corollary 1.** *For* $|z| \le 1$,

$$\mathbb{E}\left[z^{N_t(S)}\right] = \mathbb{E}\left[\exp\left(-(1 - z)\Theta(S; t)\right)\right].$$

Quite similarly, one can obtain the following general result.

**Theorem 5.** *For an arbitrary* $n \ge 1$, *disjoint subsets* $S_1, \ldots, S_n$ *of* $\mathbb{R}^d$, *and* $|z_i| \le 1$,

$$Q_{k_1, \ldots, k_n}(S_1, \ldots, S_n; t) = \mathbb{E}\left[\prod_{i=1}^{n} \frac{(\Theta(S_i; t))^{k_i}}{k_i!} \exp\left(-\Theta(S_i; t)\right)\right],$$

$$\mathbb{E}\left[\prod_{i=1}^{n} z_i^{N_t(S_i)}\right] = \mathbb{E}\left[\exp\left(\sum_{i=1}^{n} (1 - z_i)\Theta(S_i; t)\right)\right].$$

Theorem 5 easily yields both non-stationary and stationary moment characteristics of the vector $(N_t(S_1), \ldots, N_t(S_n))$.

In the rest of this subsection we assume that $E = \{1, \ldots, m\}$. Set

$$\begin{aligned}
\pi_k(t) &= \mathbb{P}(J_t = k), \\
\pi_{kl}(u, t) &= \mathbb{P}\left(J_u = k,\ J_t = l\right), \\
\nu_i(t) &= \mathbb{E}[N_t(S_i)], \quad 1 \le i \le n, \\
\nu_{ij}(t) &= \mathbb{E}\left[N_t(S_i)N_t(S_j)\right], \quad 1 \le i, j \le n.
\end{aligned}$$

**Corollary 2.** *For any* $t$ *and any* $i, j \in \{1, \ldots, n\}$,

$$\nu_i(t) = \sum_{k=1}^{m} \int_0^t \pi_k(t - u)\xi_k(\Upsilon_{-u}[S_i])\overline{F}_k(u)\, du,$$

$$\nu_{ij}(t) = \sum_{k,l=1}^{m} \int_{u,v=0}^{t} \pi_{kl}(t - u, t - v)\xi_k(\Upsilon_{-u}[S_i])\overline{F}_k(u)\, \xi_l(\Upsilon_{-v}[S_j])\overline{F}_l(v)\, du\, dv.$$

Assume now that the phase process $J$ is time-homogeneous (yielding, in particular, that $\pi_{kl}(t,\, t+s)$ does not depend on $t$). Let

$$\pi_k = \lim_{t\to\infty} \pi_k(t),$$
$$\pi_{kl}(s) = \lim_{t\to\infty} \pi_{kl}(t,\, t+s), \quad s \geq 0.$$

In this case, there exist limiting values

$$\nu_i = \lim_{t\to\infty} \nu_i(t),$$
$$\nu_{ij} = \lim_{t\to\infty} \nu_{ij}(t).$$

**Corollary 3.**

$$\nu_i = \sum_{k=1}^{m} \pi_k \int_0^\infty \xi_k(\Upsilon_{-u}[S_i])\overline{F}_k(u)\, du\,,$$
$$\nu_{ij} = \sum_{k,l=1}^{m} \int_{u\leq v} \pi_{kl}(v-u)\xi_k(\Upsilon_{-u}[S_i])\overline{F}_k(u)\xi_l(\Upsilon_{-v}[S_j])$$
$$+ \sum_{k,l=1}^{m} \int_{u>v} \pi_{kl}(u-v)\xi_k(\Upsilon_{-u}[S_i])\overline{F}_k(u)\xi_l(\Upsilon_{-v}[S_j])\overline{F}_l(v)\, du\, dv\,.$$

In the case of a Markov modulated spatial arrival process, $J$ is a time-homogeneous Markov process defined by intensities $\Lambda_i$ and probabilities $p_{ij}$, $i, j \in E$. Then the probabilities $\{\pi_k\}$ are stationary probabilities of $J$ and

$$\pi_{kl}(s) = \pi_k p_{kl}(s)\,,$$

where $p_{kl}(s) = \mathbb{P}(J_s = l \mid J_0 = k)$ are transition probabilities of $J$ that can be obtained from the corresponding Kolmogorov differential equations.

Note that, in Lemma 1 as well as in Theorem 5, it is possible to assume that each customer chooses its own route randomly that is, to assume that the group $\Upsilon_s$ is not fixed but can be regarded as a random element assigned (independently) to each arriving customer. Such a generalization is straightforward and leads to changing in the function (4.6). Evidently, it allows to model an individual choice of the route by each customer. But the details of this generalization are out of the scope of this paper.

### *4.3. The $SBMAP/G/\infty$ Model with Moving Customers*

In this subsection we generalize the results of Subsection 3.1 to the case of moving customers. Let the mapping $\Upsilon_s$ be continuous in the following sense. Denote $S_s = \Upsilon_{-s}[S]$ and $dS_s = S_s \setminus S_{s-ds}$. Suppose that

$$\lim_{\tau\to 0}\{S_s \setminus S_{s-\tau}\} = \emptyset\,, \quad \forall s > 0\,, \tag{4.7}$$

and

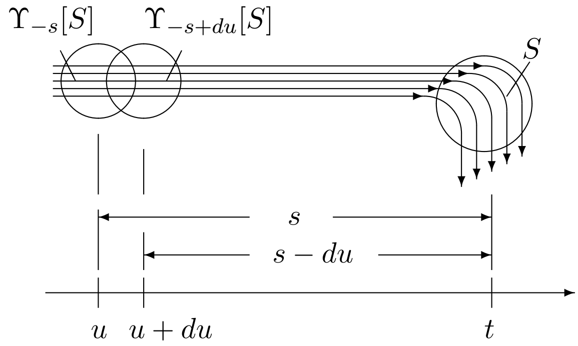
$$\Phi_{ij;n}(dS_s) = O(ds), \quad \forall i, j \in E\ n \geq 0. \tag{4.8}$$

Let $N_{u,t}(S)$ be again the number of customers who arrived in a subset $S$ until time $u$ and are still in service at time $t \geq u$. Then, with $s = t - u$, we denote

$$Q_{r;ij}(\Upsilon_{-s}[S];\, u, t) = \mathbb{P}\left(N_{u,t}(\Upsilon_{-s}[S]) = r, J_u = j \mid J_0 = i\right)$$

for the probability that $J_u = j$ and the number of $t$-resident customers in $\Upsilon_{-s}[S]$ at time $u$ is r. As usual, the entries $Q_{r;ij}(\Upsilon_{-s}[S]; u, t)$ form the matrix $Q_r(\Upsilon_{-s}[S]; u, t)$.

To describe the dynamics of the process, we observe the number of arrivals in the varying set $\Upsilon_{-\tau}[S]$ for $s \geq \tau \geq s - du$ during an infinitesimal interval of length $du$ (see Figure).




**Figure**

Let, for $t - u = s$, $\Pi_{n;ij}(S; s, du)$ be the probability that $J_u = i$ and the number of those arrivals during the interval $[u, u + du]$ that would occur in the set $S$ at time $t$ is $n$ provided that $J_{u+du} = j$, and set

$$\Pi_n(S; s, du) = (\Pi_{n;ij}(S; s, du))_{i,j \in E} .$$

Obviously,

$$\begin{aligned} \delta_{n0}\mathbf{I} + D_n(\Upsilon_{-s}[S] \cap \Upsilon_{du-s}[S])\, du + o(du) \ &\leq\ \Pi_n(S; s, du) \\ &\leq \delta_{n0}\mathbf{I} + D_n(\Upsilon_{-s}[S] \cup \Upsilon_{du-s}[S])\, du + o(du) . \end{aligned}$$

By definition of $D_n$ and according to (4.7) and (4.8),

$$\begin{aligned} D_{n;ij}(\Upsilon_{-s}[S] \cap \Upsilon_{du-s}[S]) &= \lambda_i p_i(n, j)\Phi_{ij;n}(S_s) + O(du), \\ D_{n;ij}(\Upsilon_{-s}[S] \cup \Upsilon_{du-s}[S]) &= \lambda_i p_i(n, j)\Phi_{ij;n}(S_s) + O(du) . \end{aligned}$$

Therefore,

$$\Pi_n(S; s, du) = \delta_{n0}\mathbf{I} + D_n(\Upsilon_{-s}[S])\, du + o(du) .$$

The probabilities $Q_{r;ij}(\Upsilon_{du-s}[S]; u + du, t)$ can now be expressed by

$$\begin{aligned} Q_{r;ij}&(\Upsilon_{du-s}[S]; u + du, t) \\ &= \sum_{k=0}^{r} \sum_{n=r-k}^{\infty} Q_{k;ij}(\Upsilon_{-s}[S]; u, t)\Pi_n(S; s, du) b_{r-k}(n, F(s)), \end{aligned} \tag{4.9}$$

where the probabilities $b_{r-k}(n, F(s))$ are defined in (3.1). It follows that

$$\begin{aligned} Q_r&(\Upsilon_{du+u-t}[S]; u + du, t) - Q_r(\Upsilon_{u-t}[S]; u, t) \\ &= \sum_{k=0}^{r} \sum_{n=r-k}^{\infty} Q_{k;ij}(\Upsilon_{-s}[S]; u, t)\Pi_n(S; s, du) b_{r-k}(n, F(s)) \\ &\qquad - Q_r(\Upsilon_{u-t}[S]; u, t) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial u}&Q_r(\Upsilon_{u-t}[S]; u, t) \\ &= \sum_{k=0}^{r} \sum_{n=r-k}^{\infty} Q_k(\Upsilon_{u-t}[S]; u, t) D_n(\Upsilon_{u-t}[S]) b_{r-k}(n, F(t - u)) \\ &= \ (\mathcal{Q}(\Upsilon_{u-t}[S]; u, t) * \mathcal{R}(\Upsilon_{u-t}[S]; t - u))_r\ , \end{aligned} \tag{4.10}$$

where $r \geq 0$ and $\mathcal{Q}$ and $\mathcal{R}$ are defined in Subsection 3.1. The equation (4.10) completely corresponds to the equation (3.2). The solution to (4.10) can be found similarly to (3.2), according to Theorem 1 in [6], by the following iteration algorithm [5]:

$$\mathcal{Q}^{(0)}(\Upsilon_{u-t}[S]; u, t) = \mathbf{1}\,,$$
$$\mathcal{Q}^{(i+1)}(\Upsilon_{u-t}[S]; u, t) = \mathbf{1} + \int_0^u \mathcal{Q}(\Upsilon_{s-t}[S]; s, t) * \mathcal{R}(\Upsilon_{s-t}[S]; t-s)\,ds$$

and

$$\mathcal{Q}(S; t) = \lim_{i\to\infty} \mathcal{Q}^{(i)}(\Upsilon_{u-t}[S]; u, t)\bigg|_{u=t}.$$

The equilibrium distribution exists for each stable BMAP and each service time distribution with finite mean. It is given by the probabilities

$$Q_{r;j}(S) = \lim_{t\to\infty} Q_{r;ij}(S;t) \quad \forall\, j \in E, r \geq 0\,,$$

that are independent of the initial phase $i$.

### *4.4. The $SMAPA/G/\mathbf{c}/\mathbf{c}$ System*

Similarly to the previous section, analysis of the $SMAPA/G/\mathbf{c}/\mathbf{c}$ system with moving customers can be performed by replacing the rate matrices $D^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k}}(S)$ in expression (3.6) by the rate matrices associated with $\Upsilon_{-s}[S]$, $s = t - u$. This means that, in equation (3.8), we have to replace the matrix $\mathcal{H}_{\mathbf{c}}(S;\, u, t)$ by its counterpart $\mathcal{H}_{\mathbf{c}}(\Upsilon_{-s}[S];\, u, t)$, i.e., we have to use the expressions

$$U^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k},\mathbf{r}-\mathbf{m}}(\Upsilon_{u-t}[S]\,; u, t) = D^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k}}(\Upsilon_{u-t}[S]) \prod_{c=1}^{d} b_{r_c-m_c}(l_c - k_c; F_c(t-u))\,,$$
$$V^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k}}(\Upsilon_{u-t}[S];\, u, t) = D^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k}}(\Upsilon_{u-t}[S]) \prod_{c=1}^{d} b_0(l_c - k_c; F_c(t-u)) + D^{(\mathbf{k})}_{\mathbf{l}-\mathbf{k}}(\Re \setminus \Upsilon_{u-t}[S])\,,$$

and $D^{(\mathbf{l})}_{\mathbf{0}}(\Upsilon_{u-t}[S])$, substituting them in expression (3.9) in order to define the components $(\mathcal{H})_{i,j}$ of $\mathcal{H}_{\mathbf{c}}(\Upsilon_{-s}[S];\, u, t)$.

With these changes all conclusions of Subsection 3.2 remain valid. In particular, Theorem 4 with the above replacements provides the solution of the matrix differential equation of type equal to (3.8), such that the conditional transientstate probabilities for the random vectors $\mathbf{N}_t(S)$ of customers in $S$ for an $SMAPA/G/\mathbf{c}/\mathbf{c}$ system with moving customers are obtained through equation (3.11). The equilibrium state probabilities $Q_{\mathbf{r};j}(S)$ are obtained by letting $t \to \infty$, which means that the relevant integration in the iteration scheme corresponding to (3.10) has first to be performed over $[0, t)$, i.e.,

$$\mathcal{X}_0(S;\, u, t) = \mathbf{I},$$
$$\mathcal{X}_{i+1}(S;\, u, t) = \mathbf{I} + \int_{s=0}^{t} \mathcal{X}_i(S; s, t)\mathcal{H}_{\mathbf{c}}(\Upsilon_{u-t}[S];\, s; t)\,ds \quad i \geq 0\,,$$

and then the limit

$$\lim_{t\to\infty} \mathfrak{Q}^{[\beta]}(S;\, u, t) = \lim_{t\to\infty} \mathfrak{Q}^{[\beta]}(S; 0, t)\, \mathcal{X}(S;\, u, t)$$

has to be computed. The matrices

$$Q^{(\mathbf{l})}_{\mathbf{r}}(S) = \lim_{t\to\infty} Q^{[\beta(\mathbf{l},\mathbf{r})]}(S;\, t, t), \quad \mathbf{0} \leq \mathbf{r} \leq \mathbf{l} \leq \mathbf{c}\,,$$

provide the steady-state probability matrices $Q_{\mathbf{r}}(S)$ with the components

$$Q_{\mathbf{r};i,j}(S) = \mathbb{P}\{\mathbf{N}(S) = \mathbf{r}, J = j\}, \quad \forall i \in E$$

(that do not depend on $i$) through $Q_{\mathbf{r}}(S) = \sum_{\mathbf{l}\geq\mathbf{r}} Q_{\mathbf{r}}^{(\mathbf{l})}(S)$.

## 5. DISCUSSION

Spatial arrival processes are crucial for models where the location of customers plays a significant role, for example, for mobile communication networks. Spatial versions of BMAPs [3,4] and their generalizations have been used to investigate no-waiting models for cell networks with non-moving customers [5,6]. In this paper, the same approach was generalized to take into account the movement of customers.

The results presented here are far from being complete, and there are many open problems associated with them. Let us first mention some evident steps that require more complicated notation rather than new ideas:

- a multivariate model with infinitely many servers generalizing models from Subsections 3.1 and 4.2;
- an arbitrary level dependent BMAP that does not necessarily satisfy (3.7) (cf. [16, 17]);
- joint distributions referring to several disjoint subsets $S_1, \ldots, S_n$ for models with finite number of servers considered in Subsections 3.2 and 4.3.

Considering non-trivial generalizations and problems associated with spatial models we may obey the fact that, in real systems, customers move in accordance with their individual choice. Therefore, this possibility should be reflected in a model. We have briefly mentioned about this at the end of Subsection 4.2, pointing out to the realization of random choice of the route, but there are several other approaches to do this.

As we have seen, explicit solutions were obtained above in only the case of the Cox arrival processes. It is unrealistic to hope that explicit solutions are possible for more complex cases. But it may be possible to derive differential equations for characteristics of interest. Therefore, numerical calculations become crucial for obtaining final results. Extremely complex numerical routines are typical even in case of models with non-moving customers and non-spatial arrival processes. It is necessary to mention that, for mathematical problems of this kind, the development of corresponding software seems to be a *conditio sine qua non*. Such a development will be efficient in the case when the corresponding differential (or other) equations are stated in a form convenient for numerical routines. Recall that the requirement of convenience was a basic inspiration for introducing BMAPs (see [10]). It seems that the equations presented in this paper can be effectively solved. However, serious research in this direction is necessary.

## REFERENCES


1. Baccelli F., Klein M., Lebourges M., and Zuyev S. Stochastic Geometry and Architecture of Communication Networks. *J. Telecom. Systems*, 1996. Also In: *Selected Proceedings of the Third INFOCOM Telecommunications Conference*, 1995.

2. Baccelli F., Zuyev S. Stochastic Geometry Models of Mobile Communication Networks. In: *Frontiers in Queueing, Probability and Stochastics Series*. Ed. J.H.Dshalalow. Boca Raton: CRC Press, 1997.

3. Baum D. On Markovian Spatial Arrival Processes for the Performance Analysis of Mobile Communication Networks. *Research-Rep.*, no. 98-07, University of Trier, 1998.

4. Baum D., Kalashnikov V. Spatial Generalization of BMAPs with Finite Phase Space. *J. Apl. Prob* (to be published). Also In: *Research Rep.*, no. 98-11, University of Trier, 1998.

5. Baum D. Multi Server Queues with Markov Additive Arrivals. *Proc. 3rd Int. Conf. on Matrix Analytic Methods (MAM3)*, Leuven, Belgium, 2000. Also In: *Research Rep.*, no. 99-18, University of Trier, 1999.

6. Baum D. The Infinite Server Queue with Markov Additive Arrivals in Space. *Proc. of the RareEvents'99 Int. Conf.*, Eds. V.V.Kalashnikov and A.M.Andronov, Riga Aviation University, Riga, Latvia, 1999. Also In: *Research Rep.*, no. 98-31, University of Trier, 1998.

7. Çinlar E. Markov Additive Processes I, II. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 1972, vol. 24.

8. Prabhu N. U. Markov Renewal and Markov-additive Processes—A Review and Some New Results. In: *Proc. of KAIST Math. Workshop*, Eds. B.D.Choi and J.W.Yi, vol. 6, Korea Adv. Inst. Sci. Tech., Taejon, 1991.

9. Pacheco A., Prabhu N. U. Markov-Additive Processes of Arrivals. In: *Advances in Queueing: Theory, Methods, and Open Problems*, Ed. J.H.Dshalalow, Boca Raton: CRC Press, 1995.

10. Neuts M. F. A Versatile Markovian Point Process. *J. Appl. Prob.*, 1979, 16, pp. 764–779.

11. Neuts M. F. *Structured Stochastic Matrices of M/G/1 Type and Their Applications.* New York: Marcel Dekker, 1989.

12. Neuts M. F. *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach.* Baltimore: John Hopkins, 1981. Also published by Dover Publications, New York, 1994.

13. Lucantoni D. M. New Results on the Single Server Queue with a Batch Markovian Arrival Process. *Communications in Statistics: Stochastic Models*, 7(1), pp. 1–46, Marcel Dekker Inc., 1991.

14. Breuer L. Spatial Queues with Infinitely Many Servers. *Research Rep.*, no. 99-04, University of Trier, 1999.

15. Bellman R. *Introduction to Matrix Analysis.* Philadelphia: SIAM, 1995.

16. Hofmann, J. The BMAP/G/1 Queue with Level Dependent Arrivals. *Doctoral thesis*, University of Trier, Februar 1999.

17. Hofmann J. The BMAP/G/1 Queue with Level Dependent Arrivals—An Overview. *Selected Proceedings of the 4th INFORMS Telecom. Conf.*, Boca Raton, 1998. To appear In: *Special Issue of Telecommunication Systems,* 1999.

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