*Информационные процессы, Том 1, № 2, 2001, стр. 188–198.*




# Monotone Structures. The Best Possible Bounds of Their Reliability.

**V.G.Krivoulets***, **V.P.Polesskii****
*Moscow Institute of Physics and Technology, Moscow, Russia*
*e-mail: kvg@3ka.mipt.ru*
**Institute for Information Transmission Problems, Russian Academy of Sciences*
*e-mail: poles@iitp.ru*
Received 22.08.2001

## 1. INTRODUCTION

The first systemic exposition of the monotone-structure reliability theory was made in [1]. The exposition was later improved in [2, 3]. Unfortunately, these monographs (and others) do not reflect the contemporary best possible bounds of reliability of monotone structures.

In this article we made an attempt them. The article is based on material that was published in [4] (see [5, 6, 7] also) mainly.

Currently there exist three types of the attainable bounds for the reliability of monotone structure:

- packing bounds;
- untying bounds;
- differential-untying bounds;

All of them are analytic functions of parameters of monotone structure. The bounds are exact on classes of the structures possessing given values of the parameters. In other words, they are attainable, the best possible bounds (in terms of used parameters).

Any non-attainable bounds or statistical approximations of the reliability are beyond the scope of the article.

Bounds' applications or others related topics are beyond the scope of the article.

The article deals with the monotone structure of general form. The structure is defined by any bernoulli random set $(E; p)$ and any clutter $\mathfrak{A}$ on $E$.

We do not consider any special cases, when $(E; p)$ or $\mathfrak{A}$ possess special properties; for example, when $p(e) = p$ for any $e \in E$ (the homogeneous case).

## 2. PRELIMINARIES

### *2.1. Sets and set families*

The following notations will be used: $E$ is a finite set; $F, G, \ldots$ are subsets of $E$; $\mathfrak{S}, \mathfrak{F}, \mathfrak{A}, \ldots$ are families of subsets of $E$ (set families on $E$); $F + G$ is the union of disjoint subsets $F, G$ of $E$; $\cup\mathfrak{S} = \cup\{X : X \in \mathfrak{S}\}$ is the *support* of set family $\mathfrak{S}$ on $E$;

$$\mathfrak{S}^{\triangle} = \{X : X \subseteq E \text{ and there is } A \in \mathfrak{S} \text{ such that } A \subseteq X\}.$$

A subset $F$ of $E$ is an *antiblocking set* for a set family $\mathfrak{S}$, if $F$ meets every member of $\mathfrak{S}$ on at most one element, i.e.

$$|F \cap A| \leq 1$$

for any $F \in \mathfrak{S}$;

A set family $\mathfrak{S}$ on $E$ is

1. a *packing* if $X, Y \in \mathfrak{S},\ X \neq Y \Longrightarrow X \cap Y = \emptyset$;
2. a *monotone increasing family*, if $X \in \mathfrak{S}, X \subseteq Y \subseteq E \Longrightarrow Y \in \mathfrak{S}$;
3. a *clutter* if $X, Y \in \mathfrak{A} \Longrightarrow X \nsubseteq Y$.

The clutter $\mathfrak{B}$ *dual* (or blocker of the clutter $A$) to clutter $\mathfrak{A}$ is the family of minimal sets (by inclusion) meeting every member of $\mathfrak{A}$.

### *2.2. Independent (bernoulli) random set (binomial model) and reliability*

Let $E$ be a finite set, $p : E \longrightarrow [0,1]$ and $q = 1 - p$. For any family $\mathfrak{S}$ on $E$ put

$$P(\mathfrak{S}; p) = \sum_{F \in \mathfrak{S}} p^F q^{\overline{F}}$$

where $\overline{F} = E - F,\ p^F = \prod_{e \in F} p(e),\ q^{\overline{F}} = \prod_{e \in \overline{F}} q(e)$.

The finite probabilistic space $(\Omega,\ 2^{\Omega},\ P)$, where $\Omega = 2^E$ is called the *independent (bernoulli) random set, i.r.s.* and denoted by $(E; p)$.

If $\mathfrak{S}$ is a monotone increasing family then $P(\mathfrak{S}; p)$ is called the reliability.

Set for any family $\mathfrak{F}$ on $E$

$$R(\mathfrak{F}; p) = P(\mathfrak{F}^{\triangle}; p).$$

If $\mathfrak{A}$ is the clutter of a monotone increasing family $\mathfrak{S}$ then $\mathfrak{A}^{\triangle} = \mathfrak{S}$ and we can write $R(\mathfrak{A}; p)$ instead of $P(\mathfrak{S}; p)$.

## 3. COMBINATORICS OF MONOTONE STRUCTURE RELIABILITY

### *3.1. Untying transformation of clutter support by bifurcation of element*

Let $\mathfrak{A} = \{A_1, \ldots, A_k\}$ be a clutter on $E$ and $e \in \cup\mathfrak{A}$. Set

$$\mathfrak{D} = \{A_i :\ e \in A_i,\ i \in \{1, \ldots, k\}\}.$$

We call cardinality $|\mathfrak{D}|$ of $\mathfrak{D}$ the *degree* $v(e, \mathfrak{A})$ of element $e$ in clutter $\mathfrak{A}$. Obviously $\mathfrak{A}$ is a packing iff $v(e, \mathfrak{A}) = 1$ for any $e \in \cup\mathfrak{A}$.

Let $v(e, \mathfrak{A}) > 1$ and let $\{\mathfrak{D}_1,\ \mathfrak{D}_2\}$ be any partition of $\mathfrak{D}$ onto two subclutters, i.e. $\mathfrak{D}_1 \cap \mathfrak{D}_2 = \emptyset$ and $\mathfrak{D}_1 + \mathfrak{D}_2 = \mathfrak{D}$. Put $S = \{s_1,\ s_2\}$ and let $S \cap E = \emptyset$. Put $E' = E - e + S$, $\mathfrak{A}' = \{A'_1, \ldots, A'_k\}$, where $A'_i = A_i - e + s_i$ if $A_i \in \mathfrak{D}_i,\ i = 1, 2$ and $A'_i = A_i$ if $A_i \in \mathfrak{A} - \mathfrak{D}$.

Obviously, family $\mathfrak{A}'$ is a clutter on $E'$ also and

$$v(s_i, \mathfrak{A}') = |\mathfrak{D}_i| < v(e, \mathfrak{A}).$$

Thus, in the new monotone structure (clutter $\mathfrak{A}'$ on $E'$) old elements $e$ is absent and new element $s_1,\ s_2$ have smaller degrees.

Obviously, the sequential application of these transformations leads to a packing.

The untying transformations of clutter support were introduced by Polesskii in [7].

*3.2. Clutter factor transformation*

Let $\mathfrak{S}$ be a family on $E$, $\sim$ be an equivalence on $E$ and let $\beta = \{E_j : \ j \in J\}$ be the corresponding partition on $E$. At last, let $\beta(e)$ be the equivalence class containing element $e$. For $F \subseteq E$, put

$$\beta(F) = \cup_{e \in F} \beta(e) = \{E_j : \ E_j \cap F \neq \emptyset\}.$$

The family $\{\beta(F) : \ F \in \mathfrak{S}\}$ is called the *factor-family* $\mathfrak{S}\backslash\beta$ generated by family $\mathfrak{S}$ and partition $\beta$.

Some example is the factor-family $\mathfrak{S}\backslash\beta$ obtained by identity of the elements of some subset $X \subseteq E$; here

$$\beta = \{X, \ \{e\} : \ e \in E - X\}.$$

Obviously, it is a basic case, since any factor-family can derived by successive identifies of the blocks of partition (any order of the identities can be used).

In turn, the basic case is found on the simplest case when only two elements are identified. Indeed, if we use successively the two elements identify $|X| - 1$ times for subset $X$, $|X| > 1$, we identify the elements of $X$.

If family $\mathfrak{S}$ is a clutter, then its *factor-clutter* $\widetilde{\mathfrak{S}}$ is the clutter of family $\mathfrak{S}\backslash\beta$, i.e. the collection of minimal sets (by inclusion) of $\mathfrak{S}\backslash\beta$.

*3.3. Reliability: facts, part I.*

Let $\mathfrak{A} = \{A_1, \ldots, A_k\}$ be a clutter on $E$ and $\mathfrak{B} = \{B_1, \ldots, B_l\}$ be the blocker of $\mathfrak{A}$. Then

$$R(\mathfrak{A}; p) + R(\mathfrak{B}; q) = 1, \tag{1}$$

$$R(\mathfrak{A}; p) = P(\cup_{i=1}^{k} A_i^{\triangle}; p),$$

where $P(A_i^{\triangle}; p) = p^{A_i}, \ i = 1, \ldots, k.$

*3.4. Two clutter transformation theorems*

First of all, consider the influence of untying transformation of clutter support by bifurcation of an element on the reliability.

Let $(E; p)$ be an i.r.s. and $\mathfrak{A}$ be a clutter on $E$. Consider the new i.r.s. $(E'; p')$ and clutter $\mathfrak{A}'$ obtained by the bifurcation of element $e$, where

$$p'(s_i) = p(e), \ i = 1, 2$$

and $p'(f) = p(f)$ for $f \in E - e$ (the same notations as in the section 3.1 are used).

**Clutter bifurcation theorem (CBT).**

**Theorem 1.** *Let $\mathfrak{A} = \{A_1, \ldots, A_k\}$ be a clutter on a set $E$ and $(E; p)$ be an i.r.s. Let $\mathfrak{A}' = \{A_1', \ldots, A_k'\}$ be the clutter obtained by the untying transformation of clutter support $\cup\mathfrak{A}$ by bifurcation of element $e$. Then*

$$(p')^{A_i} = p^{A_i}, \ i = 1, \ldots, k,$$

$$R(\mathfrak{A}; p) \leq R(\mathfrak{A}'; p'),$$

*and*

*(i) if there are $A_i \in \mathfrak{D}_1, \ A_j \in \mathfrak{D}_2$ such that set $A_i \cup A_j - e$ does not contain the members of $\mathfrak{A}$, and $0 < p(f) < 1$ for any $f \in E$, then*

$$R(\mathfrak{A}; p) < R(\mathfrak{A}'; p'),$$

*(ii) if for any $A_i \in \mathfrak{D}_1, \ A_j \in \mathfrak{D}_2$ set $A_i \cup A_j - e$ contains some member of $\mathfrak{A}$, then*

$$R(\mathfrak{A}; p) = R(\mathfrak{A}'; p').$$

The CBT was published in 1997 in [7].

It is wonderful that the CBT was not revealed in the theory of reliability of monotone structures. It is a matter of fact that an evident hint was contained in the pioneer Esary – Proschan's work (see [8, p.195]).

The fact is that, the CBT is only a part (one part of the two) of an old well-known McDiarmid's result [9]. The result is know as the clutter percolation theorem.

**Clutter percolation theorem (CPT).**

**Theorem 2.** *Let $\mathfrak{A}$ be a clutter on a set $E$ and $\sim$ be an equivalence on $E$ satisfying one or both of the following conditions:*

*(c) $e \sim f,\ e \neq f,\ A \in \mathfrak{A} \Longrightarrow \{e, f\} \nsubseteq A$; i.e. for any equivalence class $\beta(e)$ and any member $A$ of clutter $\mathfrak{A}$ inequality $|\beta(e) \cap A| \leq 1$ is true;*

*(c*) if $e \sim f$ and $A,\ B \in \mathfrak{A},\ e \in A - B,\ f \in B - A$, then there exists a member $D$ of clutter $\mathfrak{A}$ such that $D \subseteq A \cup B - \{e, f\}$.*

*Let $(E; p)$ be an i.r.s. such that $e \sim f \Longrightarrow p(e) = p(f)$ and let $\widetilde{p}(\beta(e)) = p(e)$.*

*Put*

$$\triangle = R(\mathfrak{A}; p) - R(\widetilde{\mathfrak{A}}; \widetilde{p}).$$

*Then, if condition (c) is satisfied, then $\triangle \geq 0$. If condition (c*) is satisfied, then $\triangle \leq 0$. Thus, if both conditions (c), (c*) are satisfied, then $\triangle = 0$. If exactly one of the conditions (c), (c*) is satisfied and $0 < p(e) < 1$ for any element $e \in E$, then $\triangle \neq 0$.*

The CPT was published by McDiarmid yet in 1981 [9].

Obviously, that the CBT is a part of the CPT. Indeed, the untying transformation of clutter support by bifurcation of element $e$ (when $e$ is replace by two new elements $s_1,\ s_2$) is a special inverse transformation to the factor clutter transformation used by McDiarmid in the CPT. It corresponds to the case, where every equivalence class contains one element, except the unique equivalence class that contains two elements $s_1,\ s_2$. In this case condition (c) is true, of course.

Polesskii rediscovered independently the CBT (one port of the CPT); but his proof original and very simple.

In fact, the CPT is a mixture of two components.

The first, "factor part" describes influence of the factor clutter transformation (satisfying condition (c*) only) on the reliability of monotone structure. This part deals with one clutter principally. It was proved in [6] also by simple arguments analogous to ones used in the proof of the CBT in [7].

The second, "untying part" describes influence of the untying transformation of clutter support by bifurcation of element on the reliability of monotone structure. Indeed, it is correctly to consider that the part (dealing with the case (c)) is obtained from the following, more general, result. Polesskii generalized the CBT on case of transformations on the supports of any clutter-family [6]. He showed that these transformations do not decrease the reliability of the corresponding clutter-sum and constructed bounds for the clutter-sum reliability. Publication [6] contains a new simple proof of the BK inequality [13, 14] also.

*3.5. Reliability: facts, II*

Obviously, that

$$R(\mathfrak{A}; p) = P\left(\bigcup_{i=1}^{k} A_i^{\triangle}; p\right) = \sum_{j=1}^{k} P\left(\mathfrak{D}_j; p\right), \tag{2}$$

where $\mathfrak{D}_1 = A_1^{\triangle},\ \mathfrak{D}_j = \overline{A_1^{\triangle}} \ldots \overline{A_{j-1}^{\triangle}} A_j,\ j = 2, \ldots, k.$

Relationship (2) is usually called *the disjoint-product formula*. It is not difficult to verify, that

$$R(\mathfrak{D}_j;p) = p^{A_j} R\left(\mathfrak{C}_j^*; q/E - A_j\right), \ j = 2, \dots, k,$$

where $\mathfrak{C}_j^*$ is the blocker of clutter $\mathfrak{C}_j$ of minimal sets (by inclusion) of family

$$\mathfrak{A}_j = \{A_i - A_j : \ i = 1, \dots, j-1\}$$

of differences of the enumerated members of $\mathfrak{A}$, and $q/E - A_j$ is the restriction of $q$ on $E - A_j$.

Therefore

$$R(\mathfrak{A};p) = p^{A_1} + \sum_{j=2}^{k} p^{A_j} R(\mathfrak{C}_j^*; q/E - A_j). \tag{3}$$

Relationships, (1) and (3) imply that

$$R(\mathfrak{A};p) = p^{A_1} + \sum_{j=2}^{k} p^{A_j} (1 - R(\mathfrak{C}_j; q/E - A_j)). \tag{4}$$

Now we have had the combinatorics to construct the best possible bounds for reliability.

## 4. PACKING BOUNDS

They are trivial. Obviously, that

$$R(\mathfrak{A}';p) \le R(\mathfrak{A};p) \le 1 - R(\mathfrak{B}';q),$$

where $\mathfrak{A} = \{A_{i_1}, \dots, A_{i_t}\}$, $\mathfrak{B}' = \{B_{j_1}, \dots, B_{j_s}\}$ are subclutters of $\mathfrak{A}$ and $\mathfrak{B}$, accordingly. If $\mathfrak{A}'$ is a packing, then

$$R(\mathfrak{A}';p) = 1 - \prod_{j=1}^{t} (1 - p^{A_{i_j}})$$

and we have a lower packing bound (the disjoint-minpaths lower bound) of the reliability in terms of a packing of members of direct clutter $\mathfrak{A}$:

$$1 - \prod_{j=1}^{t} (1 - p^{A_{i_j}}) \le R(\mathfrak{A};p). \tag{5}$$

If $\mathfrak{B}'$ is a packing, then

$$1 - R(\mathfrak{B}';q) = \prod_{j=1}^{s} (1 - q^{B_{i_j}}),$$

and we have an upper packing bound (the disjoint-mincuts upper bound) of the reliability in terms of packing of members of dual clutter $\mathfrak{B}$:

$$R(\mathfrak{A};p) \le \prod_{j=1}^{s} (1 - q^{B_{i_j}}). \tag{6}$$

The packing bounds (5), (6) are found on the monotonicity of measure, trivial side of statistic independence for monotone events (in a private case, when the clutters contain one member only), and on duality relation (1).

Both bounds (5), (6) are attainable. Attainability of (5) corresponds to case, when $\mathfrak{A}$ is a packing. Attainability of (6) corresponds to case, when $\mathfrak{B}$ is a packing.

Packing bounds (5), (6) are so trivial that nobody wishes to pretend to be their author. However, for a special case of monotone structure (two-terminal reliability) such a pretenders there exist. They are E.I.Litvak and I.A.Ushakov (see [10])); later I.A.Ushakov published a “Handbook of Reliability Engineering” in the USA [11].

## 5. UNTYING BOUNDS

The CBT means that the untying transformations of clutter support by bifurcations of its elements (changing elements of clutter support) generate a whole class of "untying" bounds.

It is necessary to emphasize that from 1981 nobody extracted any sequence from the CPT to construct bounds for the reliability of monotone structure, despite of a McDiarmid's remark (see [9]). McDiarmid himself used the CPT to investigate interrelations between reliability of random graphs and random digraphs, mainly.

He also deduced from the CPT the Harris inequality [12] (i.e. the correlation inequalities) and the BK inequality [13] (see [14] also).

### Correlation inequalities

Let $\mathfrak{S}, \mathfrak{F}$ be monotonically increasing (decreasing) events (it is essential that they both have the similar monotonicity). The following correlation inequality.

$$P(\mathfrak{S};p)P(\mathfrak{F};p) \leq P(\mathfrak{S} \cap \mathfrak{F};p) \tag{7}$$

was first proved by Harris [12].

Since them, such inequality has been reworked in more general context, by current convention it is named as the Harris-FGK inequality. At present the correlation inequalities are obtained by the FKG-inequality (see, for example [15, Chapter 6]).

The Harris inequality was reopened by Esary and Prosch [8] in the monotone-structure reliability theory.

It is necessary to note also that the correlation inequalities and characterization of statistical independence of monotone events are easily obtained from the more general (than the CBT) theorem [6]. It is necessary to use its specific case dealing with the untying transformation of two clutter supports. These simple proofs (see [6, 7]) differ from the Harris' proof [12], the Esary – Proschan's proof [8], the proof [15] using the FKG-inequality, and lastly, from the McDiarmid's proof.

Now, let us back to the untying bounds properly. As we have mentioned, they are based on the CBT. We can obtain the attainable bounds (as analytic formulas) for $R(\mathfrak{A};p)$, when almost all elements of support $\cup\mathfrak{A}$ have been untied.

### Esary–Proschan or untying-packing bounds

The extreme case is a packing. The classical Esary – Proschan upper bound

$$R(\mathfrak{A};p) \leq 1 - \prod_{i=1}^{k}(1 - p^{A_i}), \tag{8}$$

corresponds to the extreme case namely. The lower Esary – Proschan bound

$$\prod_{i=1}^{l}(1 - q^{B_i}) \leq R(\mathfrak{A};p), \tag{9}$$

follows form (1) and (8).

However, Esary and Proschan proved their bounds using the correlation inequality (7).

Up to now the theory of reliability holds that the Esary–Proschan bounds are based on the correlation inequality (7). Of course, we can deduce them from (7). However, we can not prove other untying bounds (for example the untying-antiblocking bounds) in such a way. In this case we must use the CBT. In view of the CBT there exists, we do not need any correlation inequalities to prove the bounds.

The Esary – Proschan bounds and all other untying bounds are based on the CBT. It is their natural foundation.

It is necessary to emphasize that the Esary–Proschan bounds (8), (9) are the worst untying bounds (we call them the untying-packing bounds also). Of course, we can not improve them in terms of used parameters. Really, upper bound (8) is attained, when clutter $\mathfrak{A}$ is a packing; lower bound (9) is attained, when its blocker $\mathfrak{B}$ is a packing.

**Untying-antiblocking bounds**

How can we improve the Esary–Proschan bounds? We we must stop the untying transformations in a good time, before the packing. For example, do the following.

Let $F$ be an antiblocking set for clutter $\mathfrak{A}$ and let $F$ does not belong to the clutter (for example, if $\mathfrak{A}$ is the clutter of $s$-$t$ paths in a graph, then $F$ can be the star of vertex $s$).

First of all, untie fully the member of clutter $\mathfrak{A}$ that do not meet $F$.

In the second place, untie fully the member of clutter $\mathfrak{A}$ meeting with $F$ on element $e$, besides element $e$. Do it for every element $e$ from $F_1 = F \cap (\cup\mathfrak{A})$. As a result, we obtain some clutter-flowers (see [7]). The untying bound generated by the clutter-flowers is strongly better than the Esary–Proschan bounds (8), (9).

Let $\mathfrak{A}'$ be the members of $\mathfrak{A}$ that does not meet $F$, $\mathfrak{A}''(e)$ be the members of $\mathfrak{A}'' = \mathfrak{A} - \mathfrak{A}'$ meeting $F$ on element $e$. Then

$$R(\mathfrak{A};p) \le 1 - \prod_{A\in\mathfrak{A}'}(1-p^A)\prod_{e\in F_1}\left(1-p(e)\left(1-\prod_{B\in\mathfrak{A}''(e)}(1-p^{B-e})\right)\right). \tag{10}$$

Bound (10) is strongly better than (8). From (10) and (1) we have

$$\prod_{B\in\mathfrak{B}'}(1-q^B)\prod_{e\in G_1}\left(1-q(e)\left(1-\prod_{C\in\mathfrak{B}''(e)}(1-q^{C-e})\right)\right) \le R(\mathfrak{A};p), \tag{11}$$

where $G$ is an antiblocking set for $\mathfrak{B}$ ($C \notin \mathfrak{B}$). The notations in (11) are as in (10).

We call bounds (10), (11) the *untying-antiblocking bounds*. Bounds (10), (11) are attainable also (see [7]).

## 6. DIFFERENTIAL-UNTYING BOUNDS

They are based on the untying bounds and on the disjoint-product formulas (3), (4).

### *6.1. Differential-untying packing bounds and differential-untying-antiblocking bounds*

If we estimate each reliability $R(\mathfrak{C}_j^*; q/E - A_j)$ in (3) below by means of untying-packing bound (8) or by untying-antiblocking bound ((10), we obatin the following differential-untying-packing lower bound (12) and differential-untying-antiblocking lower bound (13), accordingly:

$$p^{A_1} + \sum_{j=2}^{k} p^{A_j}\prod_{i=1}^{j-1}(1-p^{A_i-A_j}) \le R(\mathfrak{A};p), \tag{12}$$

$$p^{A_1} + \sum_{j=2}^{k} p^{A_j}\prod_{C\in\mathfrak{C}_j'}(1-p^C)\prod_{e\in F_j'}\left(1-p(e)\left(1-\prod_{D\in\mathfrak{C}_j''(e)}(1-p^{D-e})\right)\right) \le R(\mathfrak{A};p), \tag{13}$$

where $F_j$ is an antiblocking set for clutter $\mathfrak{C}_j$ ($F_j \notin \mathfrak{C}_j$); $\mathfrak{C}_j''(e)$ is the subclutter of clutter $\mathfrak{C}_j''$ meeting $F_j$ on element $e$; $\mathfrak{C}_j' = \mathfrak{C}_j - \mathfrak{C}_j''$; $F_j' = F_j \cap (\cup\mathfrak{C}_j)$.

Bound (13) is better than (11). From (12), (13) and (1) we have the following upper differential-untying bounds:

$$R(\mathfrak{A};p) \leq 1 - q^{B_1} - \sum_{j=2}^{l} q^{B_j} \prod_{i=1}^{j-1} (1 - q^{B_i - B_j}), \tag{14}$$

$$R(\mathfrak{A};p) \leq 1 - q^{B_1} - \sum_{j=2}^{l} q^{B_j} \prod_{C \in \mathfrak{D}'_j} (1 - q^C) \prod_{e \in G'_j} \left(1 - q(e) \left(1 - \prod_{D \in \mathfrak{D}''_j(e)} \left(1 - q^{D-e}\right)\right)\right), \tag{15}$$

where the notations are analogous to ones in (12), (13), accordingly.

## Differential-untying bounds of second order

There are other differential-untying bounds. Indeed, we can apply bound (12) to reliability $R(\mathfrak{C}_j; p/E - A_j)$ in (4). Then we obtain an upper bound for $R(\mathfrak{A};p)$ as a function of $(\mathfrak{A};p)$. Put $C_i^j = A_i - A_j,\ i = 1, \ldots, j-1$ and let $\mathfrak{C}_j = \{C_1^j, \ldots, C_{u(j)}^j\}$. Then for $k > 2$

$$R(\mathfrak{A};p) \leq p^{A_1} + \sum_{j=2}^{k} p^{A_j} \left(1 - p^{C_1^j} - \sum_{i=2}^{u(j)} p^{C_i^j} \prod_{s=1}^{i-1} \left(1 - p^{C_s^j - C_i^j}\right)\right). \tag{16}$$

where set $C_s^j - C_i^j$ is a difference of second order (difference of difference) $(A_s^j - A_j) - (A_i^j - A_j) = A_s^j - A_i^j - A_j$.

Bound (16) uses differences of differences in contrast to (12) using differences only. Of course, we can apply (13) instead of (12) to estimate $R(\mathfrak{C}_j; p/E - A_j)$ in (4) and obtain a better bound. Analogously if $D_i^j = B_i - B_j,\ i = 1, \ldots, j-1$ and $\mathfrak{D}_j = \{D_1^j, \ldots, D_{v(j)}^j\}$ is the clutter of minimal sets (by inclusion) of family $\mathfrak{B}_j = \{D_i^j,\ i = 1, \ldots, j-1\}$ of differences of mentioned members of blocker $\mathfrak{B}$, then we obtain for $l > 2$

$$1 - q^{B_1} - \sum_{j=2}^{l} q^{B_j} \left(1 - q^{D_1^j} - \sum_{i=2}^{v(j)} q^{D_i^j} \prod_{s=1}^{i-1} \left(1 - q^{D_s^i - D_i^j}\right)\right) \leq R(\mathfrak{A};p). \tag{17}$$

where set $D_s^j - D_i^j$ is a difference of second order of the indicated members of blocker $\mathfrak{B}$.

We call bounds (16), (17) *the differential-untying bounds of second order*.

### *6.2. Attainability of differential-untying bounds*

The differential-untying bounds are the best possible ones, generally. The problem of their attainability is interesting. It was investigated in [5] partially.

It was shown in [5] that if clutter $\mathfrak{A}$ possess the following condition

$$A_i - A_j = A_i - A_{j+1},\ 1 \leq i < j \leq k-1,$$

then the left side in the differential-untying packing bounds (12) is a two-terminal $s$-$t$ reliability for a random graph.

### *6.3. Differential-untying bounds and Oxley – Welsh bounds of homogeneous monotone structure reliability*

The differential-untying packing bounds (12) are natural generalization (and improvement) for the following Oxley–Welsh bounds [16, 17] of reliability, when i.r.s. $(E; p)$ is homogeneous, i.e. $p(e) = p$ for any element $e$ of $E$. Namely, Oxley and Welsh proved [16, 17] in 1979 the following result. Let $|A_i| = a_i,\ i = 1, \dots, k$. Then

$$\sum_{i=1}^{k} p^{a_i} q^{i-1} \le R(\mathfrak{A}; p). \tag{18}$$

Bound (18) and (1) give the upper bound

$$R(\mathfrak{A}; p) \le 1 - \sum_{j=1}^{l} q^{b_j} p^{j-1}, \tag{19}$$

where $b_i = |B_i|,\ i = 1, \dots, l$.

It is wonderful that Oxley and Welsh do not extend their bounds (18), (19) on general i.r.s. $(E; p)$ (improving them essentially at the same time). Again, wonderfully is that McDiarmid, who reviewed their work, did not notice the natural generalization.

Compare (12) and (18) in the homogeneous case. In the case (12) is

$$p^{a_1} + \sum_{j=2}^{k} p^{a_j} \prod_{i=1}^{j-1} (1 - p^{|A_i - A_j|}) \le R(\mathfrak{A}; p), \tag{20}$$

But $|A_i - A_j| \ge 1,\ i = 1, \dots, j-1$, therefore

$$\prod_{i=1}^{j-1} \left(1 - p^{|A_i - A_j|}\right) \ge q^{j-1} \tag{21}$$

and the equality in (21) takes a place iff $|A_i - A_j| = 1,\ i = 1, \dots, j-1$. Therefore, even for the homogeneous i.r.s. $(E; p)$ bound (20) generally better than Oxley–Welsh bound (18) (except one trivial case of attainability of (18) (see [16, 17]), when they are both equal).

It is natural since (20) uses additional parameters $|A_i - A_j|,\ i = 1, \dots, j-1;\ j = 2, \dots, k$ besides cardinalities $a_i,\ i = 1, \dots, k$. We can not improve (18) in terms of $a_i,\ i = 1, \dots, k$ only.

## 7. MONOTONICITY OF PACKING BOUNDS, DIFFERENTIAL-UNTYING PACKING BOUNDS, AND DIFFERENTIAL-UNTYING-ANTIBLOCKING BOUNDS

Let $L(\mathfrak{A}; p),\ U(\mathfrak{B}; q)$ be the lower bound and upper bound for the following pairs: (5) and (6); (12) and (14); (13) and (15). $L(\mathfrak{A}; p)$ is a function of $(\mathfrak{A}; p)$; $U(\mathfrak{B}; q)$ is a function of $(\mathfrak{B}; q)$.

Let $\mathfrak{A}' \subseteq \mathfrak{A},\ \mathfrak{B}' \subseteq \mathfrak{B}$. Since $L(\mathfrak{A}'; p) \le L(\mathfrak{A}; p)$ and $U(\mathfrak{B}'; q) \le U(\mathfrak{B}; q)$, then $L(\mathfrak{A}'; p)$ and $U(\mathfrak{B}'; q)$ are also a lower bound and an upper bound for reliability $R(\mathfrak{A}; p)$, respectively.

It contrast to these bounds, the untying bounds (8), (9) and (10), (11) do not possess such a monotone property. We need use all minpaths and mincuts principally in them.

The abovementioned monotonicity can be used to estimate reliability $R(\mathfrak{A}; p)$ when clutter $\mathfrak{A},\ \mathfrak{B}$ have a huge size (it is true for the network reliability measures, generally).

In order to estimate $R(\mathfrak{A}; p)$ is such a case, it is naturally to use small parts of $\mathfrak{A},\ \mathfrak{B}$ (giving a considerable deposit).

For example, it is naturally to apply the differential-untying bounds (12), (13) for subclutters of minpaths and mincuts of limited values.

Clearly, that the differential-untying bounds introduce new ideas, in particular, we can not search "the best packing bounds" (notice that such a search can be $NP$-hard according V.Raman's result) as C.J.Colbourn appeals in [18]. We can search, instead if them, "the best differential-untying bounds".

## 8. DIFFERENTIAL-UNTYING QUASIPACKING BOUNDS FOR NETWORK RELIABILITY

Work [4] is a first attempt to do this. It describes differential-untying quasipacking bounds for network reliability. If all known packing bounds (see [18]) use packing of paths and cuts, the differential-untying quasipacking bounds use "quasipackings" obtained by addition of minpaths and mincuts to the packings. The additional members are met and meet members of the packings, generally.

In [4] differential-untying quasipacking bounds for reliability of random matroid, $K$-terminal reliability, reachability of random digraph, connectedness probability of random graph and two terminal reliability improving known "packing bounds" of these characteristics are described. For example, it is easy to improve (see [4]) the modified packing lower bound for connectedness probability of homogeneous random graph

$$1 - (1 - p^{n-1})^c \leq R(G; p),$$

obtained by A.Ramanathan and C.J. Colbourn (see [18]), where $c$ is the edge connectivity of graph $G$ and $n$ is the number of vertices of $G$.

It is easy to modify (improve) the heuristic of T.K. Brecht and C.J. Colbourn for packing lower bounds of two-terminal reliability and heuristic of C.J. Colbourn at al. (L.D. Nel, N.J. Stayer, D.K. Wagner, J.G. Santhikumar) for searching efficient and good upper bounds of two-terminal reliability.

## REFERENCES


1. Barlow R.E., Proschan F. *Mathematical Theory of Reliability*, New York: Wiley, 1965.
2. Barlow R.E., Proschan F. *Statistical Theory of Reliability and Life Testing. Probability Models*. New York: Holt, Rinehart and Winston, 1975.
3. Barlow R.E., Proschan F. *Statistical Theory of Reliability and Life Testing. To Begin With*. Md: Silver Spring, 1981.
4. Krivoulets V.G., Polesskii V.P. Quasipacking Bounds for Network Reliability. *Information Processes*, 2001, vol. 1, no. 2, pp. 126–146 (in Russian, http://www.jip.ru).
5. KrivouletsV.G., Polesskii V.P. On Bounds of Reliability for Monotone Structure.*Problemy Peredachi Informacii*, 2001, vol. 37, no. 4 (in Russian).
6. Polesskii V.P. Untying of Clutter-Family Support and Its Role for Monotone-Structure Reliability Theory. *Problemy Peredachi Informacii*, 2001, vol. 37, no. 2, pp. 62–76 (in Russian).
7. Polesskii V.P. Clutter Untying, Correlation Inequalities, and Bounds for Combinatorial Reliability. *Problemy Peredachi Informacii*, 1997, vol. 33, no. 3, pp. 50–63 (in Russian).
8. Esary J., Proschan F. Coherent Structures of Non-Identical Components. *Technometrics*, 1963, vol. 5, no. 2, pp. 191–209.
9. McDiarmid C. General Percolation and Random Graphs. *Adv. Appl. Prob.*, 1981, vol. 13, pp. 80–70.
10. Rainshke K., Ushakov I.A. *Bounds of System Reliability Using Graphs*. Moscow: Radio i Sviaz, 1988 (in Russian).
11. *Handbook of Reliability Engineering*. New York.: Wiley, 1994.
12. Harris T.E. A Lower Bound for the Critical Probability in a Certain Percolation Process. *Proc. Cambrigde Philosophical Society*, 1960, vol. 56, pp. 13–20.

13. Van den Berg J., Kesten H. Inequalities with Application to Percolation and Reliability, *J. Appl. Prob.*, 1985, vol. 52, no. 3, pp. 553–579.

14. Grimmett G.R. *Percolation Theory*. Second edition. A Series of Comprehensive Studies in Math. Berlin: Springer, 1999.

15. Alon N., Spencer J.H., Erdos P. *The Probabilistic Method*. New York: Wiley, 1992.

16. Oxley J.G., Welsh D.J.A. On Some Percolation Results of J.M. Hammersley. *J. Appl. Prob.*, 1979, vol. 16, pp. 527–540.

17. Oxley J.G., Welsh D.J.A. The Tutte Polynomial and Percolation. *Graph Theory and Related Topics*, 1979, pp. 329–339.

18. Colbourn C.J. *The Combinatorics of Network Reliability*. Oxford: Oxford Univ. Press, 1987.

*This paper was recommended for publication by N.A.Kuznetsov, a member of the Editorial Board*