*Информационные процессы, Том 1, № 2, 2001, стр. 199–203.*




# What is the Theory of Bounds for Network Reliability?

**V.G.Krivoulets***, **V.P.Polesskii****

*Moscow Institute of Physics and Technology, Moscow, Russia*
*e-mail: kvg@3ka.mipt.ru*
**Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia*
*e-mail: poles@iitp.ru*
Received August 21, 2001

## 1. NO IDENTIFICATION

At present "the theory of bounds for network reliability" is absorbed by a more wide theory of network reliability (in one's turn, by a theory of reliability). It is not an independent subject.

In spite of monographs on reliability theory (see, for example, [1,2,3,4]) pay some attention to bounds of network reliability, it is difficult to realize the subject clearly. Nobody describes it systematically. Moreover, the subject was never a whole object for independent investigation. Of course, presentations there exist (see, for example, [4]).

Recently we made an attempt to give [5] a general look on the subject. Here are its essential features.

## 2. SUBJECT

The subject of the research of the theory of bounds for network reliability is the bounds of "reliabilities" of various random discrete structures (monotone (coherent) system, shellable simplicial complex, semilattice, lattice, matroid, graph, etc.) that generalize measures of network reliability.

Indeed, the bounds of the "reliabilities" are bounds of the measures simultaneously.

## 3. METHODS

The methods of the theory of bounds of the network reliability are methods of theories of these random discrete structures.

## 4. QUOTATIONS

Let us cite C.J.Colbourn [2] and D.R.Shier [3] to clarify stated above. C.J.Colbourn writes (see [2, Preface]): "Understanding a problem involves extracting the combinatorics at the heart of the problem. Understanding network reliability involves extracting the combinatorics of network reliability. What is "the combinatorics of network reliability?" A cursory examination reveals classical theorems on connectivity and cutsets, and basic results in combinatorial enumeration. A more careful examination, however, reveals close connections with external set theory, matroid theory, and polyhedral combinatorics, to name just a few areas."

D.R.Shier (see [3, Preface]) repeats: "... there has been in the subject because of the variety of discrete, combinatorial, and algebraic mathematics that can be found lurking just undernearth this practical veneer ... various types of algebraic objects (partially ordered sets, lattices, semilattices, and spaces of polynomials) emerge to play their roles in simplifying, uniting, and comprehending the nature of network reliability problem ..."

## 5. KEY FOR THE THEORY: THE BEST POSSIBLE BOUNDS

Perhaps the most known bounds for network reliability are the Esary–Proschan bounds [6]. They are beautiful representatives of the bounds playing (on our opinion) a key role for the theory of bounds of network reliability.

More exactly, we suppose that the fundamental type of the bounds are the analytic functions of parameters of the abovementioned random discrete structures, that are exact on classes of the structures possessing given values of the parameters.

In other words, they are attainable, the best possible bounds (in terms of used parameters).

Why are they key?

On our opinion, the attainable bounds of network reliability is a constructive path to realize how closely we can approach to the exact value of the network reliability (recall that calculation of every measure of network reliability is the $NP$-hard algorithmical problem) by means of efficiently computable bounds.

Of course, every such an attainable bound possess its own specific weakness. Speculating on the weakness we can easy construct "practical" examples, where the bound is bad.

However, the bounds can not be improved in terms of used parameters and we can not improve them "mathematically" without utilization of some additional information, because of their attainability. Practically we can, mathematically we can not. Therefore, the bounds that are more closely to the exact value of corresponding reliability measure must use other parameters or attract other discrete structure (possessing more wealthy properties).

Thus, we need to attract more and more complex mathematical objects, drawing the necessary information (their concepts (parameters) and methods) for constructing new bounds.

It is very important to emphasize that we are directed by a clear goal: constructing the attainable bounds in terms of parameters of these objects.

It is necessary to note that demand for efficient calculation of the attainable bounds leads to considerable decreasing the attracted parameters. The demand leads also to considerable decreasing the class variety on which the bounds are attained, since the reliability of these extreme constructions must be calculated efficiently.

Revealing more and more accurate efficiently calculated bounds has a general-mathematical (even philosophic) interest since it is some constructive way to realize the $NP$-hard problem.

Besides, it has undoubted practical interest motivated by the problem of synthesis of reliable network topology. We read (see [2, Concluding Remarks]): "Perhaps more important is to develop better combinatorial tools which enable us to compute accurate bounds efficiently."

## 6. THE ATTAINABLE BOUNDS AND OTHER ASPECTS OF THE THEORY OF NETWORK RELIABILITY

The attainable bounds are a skeleton, foundation for the theory of network reliability simultaneously. They lie in the basis of its results, give its definite form.

Let us demonstrate it in brief.

### *6.1. Other bounds*

**Non-attainable bounds**

The non-attainable bounds that are analytic functions of parameters of networks are obtained by ruding the attainable bounds. I.e., the attainable bounds are generating, mother ones.

A vivid example is the following Lomonosov–Polesskii lower bound [7] for the connectedness probability (all terminal reliability) $R(G;p)$ of random homogeneous graph $(G;p)$ generated by a graph $G$:

$$R(G;p) \leq (1 - q^{n-1})^{\frac{2m}{n}}, \tag{1}$$

where $n, m$ are the numbers of vertices and edges of $G$, accordingly.

**Algorithmical bounds**

The algorithmical bounds addmitting to calculate them for every given network by some algorithm use the attainable bounds and/or their crude form (roughening). A good example is algorithmical bounds using the factoring method.

*6.2. Efficient calculation of reliability for classes of random discrete structures*

We mean such set families as packings, parallel-serial structures, direct sum of matroids, complete graphs, complete bipartite graphs, acyclic graphs, etc. (see [4]).

They are the classes on which the attainable bounds are attained.

*6.3. Transformations preserving, increasing/decreasing the reliability*

The current state of network reliability shows that sequence of such transformations can directly lead to the extreme constructions (see above). For example, the sequence of the untying transformation of clutter support by bifurcation of its elements leads to packings [8], on which the well-known Esary–Proschan bounds [6] are attained.

Besides, it is possible a well-timed stop of transformation sequence, when we've obtained some structure whose reliability can be calculated (or obtained explicitly as some analytic formula) efficiently (see [3, 4, 8], for example).

*6.4. Statistical approximation of the reliability*

Generally, statistical methods of approximation of the measures of network reliability use their bounds in order to decrease the size of sampling space.

For example, the D.Karger's proof [9] of efficiency of approximation (with a preassigned accuracy) for the all terminal reliability uses some efficiently calculated and attainable bounds for the reliability.

*6.5. Asymptotic properties of the random discrete structures*

They are the limits of probabilities of some events (for example, connectivity). Also we can be interested in the asymptotic distributions of random variables (for example, the number of connectedness components of a random graph), when parameters of the structures tend to infinity.

Some asymptotic results can be obtained without the Erdos' probabilistic method using attainable bounds only. For example, let $(K_n;p)$ be the random graph generated by complete homogeneous graph $K_n$ with $n$ vertices (any edge of $K_n$ fails independently with the same probability $q = 1 - p$). P.Erdos and A.Renyi proved that if

$$np - \ln n \longrightarrow \infty \tag{2}$$

for $n \longrightarrow \infty$, then random graph $(K_n;p)$ is almost sure connected (a.s.), i.e., $R(K_n;p) \longrightarrow 1$ for $n \longrightarrow \infty$.

The best possible bound of the reliability $R(M;p)$ of random matroid $(M;p)$

$$(1 - q^{\mu})^r \leq R(M;p) \tag{3}$$

established by Polesskii [10] admits to generalize the Erdos–Renyi result on random matroids. Here $M$ is a matroid on a set $E$, $r$ is the rank of matroid $M$, $\mu$ is the largest number of disjoint bases of $M$ (analogously, every element of $E$ fails with the same probability $q = 1 - p$).

Let $\mu = \mu(r),\ p = p(r)$ and $\mu \longrightarrow \infty,\ p \longrightarrow 0$ for $r \longrightarrow \infty$.

It follows from (3) that if

$$p\mu - \ln r \longrightarrow \infty \tag{4}$$

for $r \longrightarrow \infty$, then random matroid $(M; p)$ contains a.s. a base of $M$, i.e., $R(M; p) \longrightarrow 1$ for $r \longrightarrow \infty$.

It is a well–known fact that $\mu \geq \lfloor c/2 \rfloor$ for the graphs, where $c$ is the edge connectivity of $G$. Therefore we have for random graphs $(G_n; p)$ ($G_n$ is a graph with $n$ vertices and edge connectivity $c = c(n)$) the following sequence. If

$$pc - 2\ln r \longrightarrow \infty \tag{5}$$

for $r \longrightarrow \infty$ $(r = n - 1)$, then random graph $(G_n; p)$ is a.s. connected.

For complete graph $K_n,\ \mu = \lfloor n/2 \rfloor$ and we obtain the following sufficient condition for connectivity of complete random graph $(K_n; p)$:

$$np - 2\ln n \longrightarrow \infty$$

for $n \longrightarrow \infty$. Of course, it is worse than (2), but 5) we deal with a more wide class of random graphs. Besides, in view of the attainability of (3), we can not weaken the condition (the same is true for the random matroids).

## 7. CONCLUSION

Constructing the best possible bounds for reliabilities of various random discrete structures has a great value. The problem is open. It is naturally to start with the simplest random discrete structure, i.e., the monotone (coherent) structure.

The authors made an attempt to describe the current state of the subject in [5] (see [11] also).

Despite of evident simplicity of the random monotone structure, realizing its the best possible bounds met a lot of obstacles. It is not wonderful since the bounds are key, essential properties.

## REFERENCES

1. Barlow R.E., Proschan F. *Statistical Theory of Reliability and Life Testing, to Begin With*. Md: Silver Spring, 1981.
2. Colbourn C.J. *The Combinatorics of Network Reliability*. Oxford: Oxford University Press, 1987.
3. Shier D.R. *Network Reliabilty and Algebraic Structures*. Oxford: Claredon Press, 1991.
4. Ball M.O., Coulborn C.J., Provan J.S. Network Reliability. In: *Handbooks in OR and MS*, Chapter 11. New York: Elsevier, 1995.
5. Krivoulets V.G., Polesskii V.P. Quasipacking Bounds for Network Reliability. *Information Processes*, 2001, vol. 1, no. 2, pp. 126–146 (in Russian, http://www.jip.ru).
6. Esary J.D., Proschan F. Coherent Structures of Non-Identical Components. *Technometrics*, 1963, vol. 5, no. 2, pp. 191–209.
7. Lomonosov M.V., Polesskii V.P. An Upper Bound for the Reliability of of Information Networks. *Problemy Peredachi Informacii*, 1971, vol. 7, no. 4, pp. 78–81 (in Russian).
8. Polesskii V.P. Clutter Untying, Correlation Inequalities, and Bounds for Combinatorial Reliability. *Problemy Peredachi Informacii*, 1997, vol. 33, no. 3, pp. 50–70 (in Russian).
9. Karger D.R. A Randomized Fully Polynomial Time Approximation Scheme for the All Terminal Network Reliability Problem. *Proc. of the 37th Annual Symposium on the Foundations of Computer Science*, 1996, pp. 1–13.

10. Polesskii V.P. The Best Possible Bounds for Full Rank Probability of a Random Submatroid. *Problemy Peredachi Informacii*, 1995, vol. 31, no. 4, pp. 81–99 (in Russian).

11. Krivoulets V.G., Polesskii V.P. Monotone Structures. The Best Possible Bounds of Their Reliability. *Information Processes*, 2001, vol. 1, no. 2, pp. 189–199 (in Russian, http://www.jip.ru).

*This paper was recommended for publication by N.A.Kuznetsov, a member of the Editorial Board*