Информационные процессы, Том 2, № 2, 2002, стр. 157–160.

KALASHNIKOV MEMORIAL SEMINAR

On Queueing Networks with Signals

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Received October 14, 2002

Abstract—G-networks with ordinary (positive) customers and signals are considered under the assumption that signal processing (activation) requires a random amount of time. Activated signals either move a customer from the node they arrive to another node or kill a positive customer. For a network in which positive customers are processed by a single server at every node and for a symmetrical network, the stationary state distributions are expressed in product forms.

1. INTRODUCTION

We consider G-networks ([1]) with M nodes, positive customers, and signals. External arrival flows of positive customers and signals (from node 0) are independent Poisson processes. The rate of external arrival of positive customers at node i is denoted by λ_{0i}^{+} and that of signals by λ_{0i}^{-}.

The service of a positive customer is completed at node i with probability $\mu_i^{+}(k)\Delta + o(\Delta)$ in a time interval $(t, t+\Delta)$, provided there were k positive customers at this node at instant t. Upon completion of service at node i, a positive customer passes to node j with probability p_{ij}^{+} as a positive customer, with probability p_{ij}^{-} as a signal, and with probability $p_{i0} = 1 - \sum_{j=1}^{M}(p_{ij}^{+} + p_{ij}^{-})$ quits the network.

Every signal is processed (activated) for a random time. With probability $\mu_i^{-}(n)\Delta + o(\Delta)$ a signal arriving at node i is processed in a time interval $(t, t+\Delta)$, provided there were n non-activated signals at this node at instant t. Upon completion of the activation period, a signal

- with probability q_{ij}^{+} moves a positive customer from node i to node j and retains him as a positive customer (in this case, a signal acts as a trigger [1]), or

- with probability q_{ij}^{-} moves a positive customer from node i to node j and retains him as a signal, or

- with probability q_{i0} kills a positive customer at node i and he vanishes (in this case, the signal acts as a negative customer [1]).

If there are no positive customers at node i, an activated signal at the node simply disappears.

A G-network with instantaneous signal activation is studied in [1]. An analogous G-network with random signal activation is considered in [2] in terms of quasi-reversibility. But the final formulas in [2] for product form solution are erroneous.

In this paper, the solution for a network in which positive customers are processed by a single server is derived in product form. Moreover, the stationary state distribution for the general Markov-type service of positive customers in a symmetrical network is expressed in product form too.

2. EQUILIBRIUM EQUATIONS

Introducing matrices $P^{+}, P^{-}, Q^{+}, Q^{-}$ with elements $p_{ij}^{+}, p_{ij}^{-}, q_{ij}^{+}, q_{ij}^{-}$, respectively, $i, j = \overline{1, M}$, let us take $P = P^{+} + P^{-}$ and $Q = Q^{+} + Q^{-}$.

The stochastic behavior of the queueing network can be described by a Markov process $\{X(t),\ t \geq 0\}$ with state space

$$\mathcal{X} = \{((k_1, n_1), (k_2, n_2), \ldots, (k_M, n_M)),\ \ k_i \geq 0,\ \ n_i \geq 0,\ \ i = \overline{1, M}\}.$$

The state $((k_1, n_1), (k_2, n_2), \ldots, (k_M, n_M))$ means that at any instant there are k_1 (positive) customers and n_1 (non-activated) signals at node i, k_2 customers and n_2 signals at node 2, ..., and, finally, k_M customers and n_M signals at node M.

Introducing vectors $\vec{k} = (k_1, k_2, \ldots, k_M)$ and $\vec{n} = (n_1, n_2, \ldots, n_M)$, let us take $(\vec{k}, \vec{n}) = ((k_1, n_1), (k_2, n_2), \ldots, (k_M, n_M))$. We also introduce a vector $\vec{e}_i$ with ith component equal to 1 and other components equal to 0. We also use the notation $\lambda_0^+ = \sum\limits_{i=1}^{M} \lambda_{0i}^+$ and $\lambda_0^- = \sum\limits_{i=1}^{M} \lambda_{0i}^-$. Let $p(\vec{k}, \vec{n})$ denote the stationary probability of the state $(\vec{k}, \vec{n})$. If the stationary distribution $\{p(\vec{k}, \vec{n}),\ \vec{k}, \vec{n} \geq \vec{0}\}$ of the process $\{X(t),\ t \geq 0\}$ exists, then the following system of equilibrium equations holds:

$$p(\vec{k}, \vec{n})\big(\lambda_0^+ + \lambda_0^- + \sum_{i=1}^{M} \mu^+(k_i)(1 - p_{ii}^+) + \sum_{i=1}^{M} \mu^-(n_i)\big) =$$

$$= \sum_{i=1}^{M} p(\vec{k} - \vec{e}_i, \vec{n}) \lambda_{0i}^+ u(k_i) + \sum_{i=1}^{M} p(\vec{k}, \vec{n} - \vec{e}_i) \lambda_{0i}^- u(n_i) +$$

$$+ \sum_{i=1}^{M} p(\vec{k} + \vec{e}_i, \vec{n}) \mu_i^+(k_i + 1) p_{i0} + \sum_{i=1}^{M} p(\vec{k} + \vec{e}_i, \vec{n} + \vec{e}_i) \mu_i^-(n_i + 1) q_{i0} +$$

$$+ \sum_{i=1}^{M} p(\vec{k}, \vec{n} + \vec{e}_i) \mu_i^-(n_i + 1)(1 - u(k_i)) +$$

$$+ \sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M} p(\vec{k} + \vec{e}_i - \vec{e}_j, \vec{n}) \mu_i^+(k_i + 1) p_{ij}^+ u(k_j) +$$

$$+ \sum_{i=1}^{M} \sum_{j=1}^{M} p(\vec{k} + \vec{e}_i, \vec{n} - \vec{e}_j) \mu_i^+(k_i + 1) p_{ij}^- u(n_j) +$$

$$+ \sum_{i=1}^{M} \sum_{j=1}^{M} p(\vec{k} + \vec{e}_i - \vec{e}_j, \vec{n} + \vec{e}_i) \mu_i^-(n_i + 1) q_{ij}^+ u(k_j) +$$

$$+ \sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M} p(\vec{k} + \vec{e}_i, \vec{n} + \vec{e}_i - \vec{e}_j) \mu_i^-(n_i + 1) q_{ij}^- u(n_j) +$$

$$+ \sum_{i=1}^{M} p(\vec{k} + \vec{e}_i, \vec{n}) \mu_i^-(n_i) q_{ii}^- u(n_j), \quad \vec{k}, \vec{n} \in \mathcal{X}, \tag{1}$$

where $\mu_i^+(0) = 0,\ \mu_i^-(0) = 0$ and $u(x)$ is a unit Heavyside function.

3. SOLUTION IN PRODUCT FORM

It is not possible to find the general product-form solution of the system of equations (1). Nevertheless, solutions for two important cases are given below.

Single server processing of positive customers. First consider a network in which positive customers are served at every node by a single server and the service time at node i is exponentially distributed with parameter μ_i^+. Therefore

$$\mu_i^+(k_i) = u(k_i)\mu_i^+, \;\; i = \overline{1, M}. \tag{2}$$

Let us introduce λ_i^+ and λ_i^- which are defined by the system of nonlinear equations

$$\lambda_i^+ = \lambda_{0i}^+ + \sum_{j=1}^{M} q_j(\mu_j^+ p_{ji}^+ + \lambda_j^- q_{ji}^+),$$

$$\lambda_i^- = \lambda_{0i}^- + \sum_{j=1}^{M} q_j(\mu_j^+ p_{ji}^- + \lambda_j^- q_{ji}^-), \;\; i = \overline{1, M}, \tag{3}$$

where $q_i = \lambda_i^+/(\lambda_i^- + \mu_i^+)$. Let us also take $\rho_i^-(j) = \lambda_i^-/\mu_i^-(j)$.

As in [1], we can prove that there exists a unique positive solution $\lambda_i^+, \; \lambda_i^-, \;\; i = \overline{1, M}$, of the system of equations (3).

The following theorem holds.

Theorem 1. *If the matrices P and Q are irreducible, condition (2) holds, and a unique positive solution to equations (3) exists such that $\lambda_i^+, \; \lambda_i^-, \;\; i = \overline{1, M}$,*

$$\lambda_i^+ < \lambda_i^- + \mu_i^+,$$

$$G_i = \sum_{n_i=0}^{\infty} \prod_{j=1}^{n_i} \rho_i^-(j) < \infty, \;\; i = \overline{1, M}, \tag{4}$$

then the Markov process $\{X(t), \; t \geq 0\}$ is ergodic and its stationary distribution is represented in a product form as

$$p(\vec{k}, \vec{n}) = \prod_{i=1}^{M} p(k_i, n_i), \tag{5}$$

where

$$p(k_i, n_i) = (1 - q_i) q_i^{k_i} G_i^{-1} \prod_{j=1}^{n_i} \rho_i^-(j), \;\; k_i, n_i \geq 0, \tag{6}$$

and $\prod_{j=1}^{0} \equiv 1$.

This theorem corrects the formulas derived in [2].

Symmetrical network. We consider the network described in the introduction with

$$p_{ij}^+ = q_{ij}^+, \; p_{ij}^- = q_{ij}^-, \; p_{i0} = q_{i0}, \;\; i, j = \overline{1, M}. \tag{7}$$

In this case, it is called a symmetrical network.

Let us introduce $\lambda_i^+, \lambda_i^-, \;\; i, j = \overline{1, M}$ and define them by the system of linear equations

$$\lambda_i^+ = \lambda_{0i}^+ + \sum_{j=1}^{M} \lambda_j^+ p_{ji}^+, \;\; i, j = \overline{1, M},$$

$$\lambda_i^- = \lambda_{0i}^- + \sum_{j=1}^{M} \lambda_j^+ p_{ji}^-, \quad i,j = \overline{1,M}. \tag{8}$$

If the matrix P is irreducible, the system of equations (8) has a unique positive solution for λ_i^+, λ_i^-, $i = \overline{1,M}$.

Furthermore, let us take $q_i(j) = \lambda_i^+/(\lambda_i^- + \mu^+(j))$.

The following results hold.

Theorem 2. *If equalities (7) hold, the matrix P is irreducible, and the following conditions hold:*

$$F_i = \sum_{k_i=0}^{\infty} \prod_{j=1}^{k_i} q_i(j) < \infty, \quad G_i = \sum_{n_i=0}^{\infty} \prod_{j=1}^{n_i} \rho_i^-(j) < \infty, \;\; i = \overline{1,M}, \tag{9}$$

then the Markov process $\{X(t),\ t \geq 0\}$ is ergodic and its stationary distribution is expressed by the product form

$$p(\vec{k},\vec{n}) = \prod_{i=1}^{M} p(k_i,n_i), \tag{16}$$

where

$$p(k_i,n_i) = F_i^{-1} G_i^{-1} \prod_{j=1}^{k_i} q_i(j) \prod_{l=1}^{n_i} \rho_i^-(l), \quad k_i,n_i \geq 0. \tag{10}$$

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