Информационные процессы, Том 2, № 2, 2002, стр. 161–163.

Geometric Sums in Reliability Evaluation of Regenerative Systems

J.-L. Bon

Universite des Sciences et Technologies de Lille, Eudil,
59655 Villeneuve d'Ascq, France
email: jean-louis.bon@eudil.fr
Received October 14, 2002

Abstract—Various fields of applied probability are concerned by geometric sums. We are here interested in the reparable systems reliability theory. The distribution of the sojourn time in the perfect state can be decomposed in successive i.i.d. cycles, the number of cycles having a geometric distribution. It is usual to approximate the lifetime by the random exponential variable with the same mean. We propose to bound the error improving some classical results.

1. FROM SYSTEM RELIABILITY TO GEOMETRIC SUM

For a large and highly reliable system the cumulative time spent without failure can be seen as a reasonable approximation of the system lifetime. Such an approximation is very popular because of its pessimistic aspect. It is based on the regenerative property of the process (see, for example, Solovyev (1983)). All what we need is the decomposition of the lifetime as a geometric sum of i.i.d. positive random variables. It is clearly the case when the dynamics of the system until its failure is a succession of independent cycles and the number ν of cycles is geometric.

$$P(\nu = k) = qp^{k-1},\ (q = 1 - p).$$

The probability q is nothing else that the probability of an unsuccessful restoration during a cycle. The length of the ith cycle is a random continuous variable X_i with distribution F. All these random variables are supposed independent. Some refinements can be made to take into account the particularity of the last cycle but the most difficult problem is the evaluation of the geometric sum distribution

$$W(x) = P(\sum_{k=1}^{\nu} X_i \leq x) = q \sum_{k=1}^{\infty} p^{k-1} F_k(x) \tag{1}$$

where F_k is the k-fold convolution of F. Such geometric sums appear naturally in a variety of applications and therefore have been in the focus of numerous works (see Kalashnikov (1997) and references therein).
A closed form for W is usually not available and we have to get some approximations. In reliability theory, for very small values of q, the following approximation is often used

$$W(x) = \exp\left(-\frac{qx}{E(X)}\right). \tag{2}$$

This approximation is justified by the Renyi Theorem which gives the distribution convergence when q is vanishing for each fixed x. But the exponential approximation has a good accuracy in other cases and therefore is very popular among engineers. Our aim is to propose a new family of bounds of the error from a Chen-Stein method approach. This study has been started from an idea of V.V. Kalashnikov and the results are dedicated to his memory.

2. A CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTIONS

All what follows is based on the next characterization of the exponential distribution. Let T be a random variable having the distribution function W and the reliability $\overline{W}(x) = 1 - W(x)$.

Lemma 1. *A random positive variable T has the exponential distribution E_λ if and only if*

$$E(\lambda h(T) - h'(T)) = 0 \tag{3}$$

for any non-negative function h satisfying the condition $|h'| < 1$ on $[0, \infty)$ and such that $h(0) = 0$.

From this, one can see that the quantity $E(\lambda h(T) - h'(T))$ can be viewed as a measure of closeness between W and E_λ. Let us focus on the difference at the point $a \geq 0$ and denote:

$$\Delta(a) = E_\lambda(a) - W(a). \tag{4}$$

Lemma 2. *Let fix $a \geq 0$, there exists a function h_a such that,*

$$\Delta(a) = E(\lambda h_a(T) - h_a'(T)). \tag{5}$$

is valid for any continuous positive random variable T. The function may be equal to :

$$h_a(u) = \frac{e^{-\lambda a}}{\lambda}(e^{\lambda \min(u,a)} - 1) \tag{6}$$

The usefulness of this result clearly appears in the case where no explicit form of the distribution W of T is available but a special structure of T permits to estimate (or approximate) $\Delta(a)$. It is the case when T is a geometric sum of i.i.d. random variables X_i,

$$T = \sum_{k=1}^{\nu} X_i. \tag{7}$$

Let note that W has a renewal property $W = qF + pF * W$ which can be seen in terms of variables (see Kalashnikov (1997)). Let $\stackrel{d}{=}$ denote the equality in distribution. There exist T' such that $T \stackrel{d}{=} X + \delta T'$, where T and T' have the same distribution, δ is Bernouilli and all these involved random variables are independent. Clearly $E(T) = E(\nu)E(X) = 1/\lambda$ and the distribution W of T is converging to E_λ when q is vanishing. In 1990, Brown gave some bounds to evaluate this convergence. Others works were following in the same direction as Bon and Kalashnikov (1999), Cai (1999) , Willmot and Lin (2001), $\cdots$).

3. NEW BOUNDS OF GEOMETRIC SUMS

Theorem 1. *Let $X_1, X_2, \ldots$ be i.i.d. positive random variables with the same distribution as X and ν a geometric random variable independent from X_i with*

$$P(\nu = k) = q(1-q)^{k-1}, \quad k \geq 1.$$

The geometric sum $\sum_{n=1}^{\nu} X_n$ is bounded with

$$P\left(\sum_{n=1}^{\nu} X_n \geq u\right) \leq \frac{1}{1 - q\, e^{-\lambda q u}} \left(e^{-\lambda q u} + (1 - e^{-\lambda q u})\overline{F}(u) + \frac{\lambda^2 q}{2} E(X^2)\right). \tag{8}$$

The proof is based on a local expansion of the function h in the expression

$$\lambda h(V) - h'(V) \stackrel{d}{=} \lambda h(qX + \delta V') - h'(V), \tag{9}$$

where $V = qT$ and $V' = qT'$. Some technical manipulations are needed. The expression of this upper bound can be changed with the known parameters. For example, only the moments of F may be known and a majorisation of $\overline{F}(u)$ must be used. It is useful to appreciate the comparison with the well-known bounds of Brown (1990)

$$P\left(\sum_{n=1}^{\nu} X_n > u\right) \leq \frac{1}{p} e^{-\lambda q u/p} + \frac{\lambda^2 q}{2p^2} E(X^2) \tag{10}$$

This comparison has been studied in various situations and the refinement is not uniform. Numerical simulations have given a significate improving for a large variance of F. In the same way, the Chen-Stein method provides a family of lower bounds and we propose here one of them.

Theorem 2. *Let $X_1, X_2, ...$ and ν as above. The geometric sum may be bounded with*

$$P\left(\sum_{n=1}^{\nu} X_n > u\right) \geq \frac{1}{p}\left(e^{-\lambda q u} - q - \lambda^2 q E(X^2) F(u) + \lambda q e^{-\lambda q u} F(u)\right). \tag{11}$$

Simulations have been made to appreciate the accuracy. As for the upper bound, comparing to the Brown bound (1990),

$$P\left(\sum_{n=1}^{\nu} X_n > u\right) \geq \exp\left(-\frac{q}{p}(\lambda u + \lambda^2 E(X^2) - 1)\right) \tag{12}$$

shows that the improving is not uniform. Nevertheless, these new bounds are useful as we can estimate their terms more accurately if necessary, especially, for small values of u.

REFERENCES

1. Bon, J.L. and Kalashnikov, V. V. Bounds for Geometric Sums used for Evaluation of Reliability of Regenerative Models. *J. Math. Sciences*, 1999, *93*(4), 486–510.
2. Brown, M. Error bounds for exponential approximations of geometric convolutions. *Ann. Proba.*, 1990, *18*, 1388–1402.
3. Cai, M. A unified approach to the study of tail probabilities of compound distributions *J.Appl. Prob.*, 1999, *36*, 1058–1073.
4. Kalashnikov, V. *Geometric Sums: Bounds for Rare Events with Applications*. Kluwer Academic Publishers, 1997.
5. Solovyev A.D. *Mathematical Problems in relliability theory*. ed. Gnedenko, Radio i Sviaz, 1983 (in russian).
6. Willmot G. and Lin X. *Lundberg Approximations for Compound Distributions with Insurance Applications*. Springer, 2001.