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The Help of Regenerative and Semi-Regenerative Methods for Studying a Preventive Maintenance Model

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The subject of this talk has been chosen to thank Vladimir Kalashnikov for introducing us to regenerative and semi-regenerative methods. We will illustrate the help that these methods can offer in reliability through the study of a preventive maintenance model. But, as it will be seen in the following, many ideas can be transposed to other models.

1. THE PREVENTIVE MAINTENANCE MODEL

We consider an item the behavior of which is described by a stochastic process $(\Phi_t)_{t\geq 0}$ taking values in a finite states space E. Let (E_w, E_f) be a partition of the states space E : the states of E_w are the working states of the item, the states of E_f are the failure states. If no preventive maintenance was planned, the item would evolve until it reaches a failure state, the transition rates from working states to other working states or to failures states being constant. Indeed a supervisation by way of inspections occurs as follows : when an inspection takes place, if the item is observed in a state which is not too damaged, then no service is rendered and the date of the next inspection is chosen according to a distribution which depends on the observed state. Il the item is observed in a too damaged state, a service (named preventive maintenance) is rendered. When the item failed, it is repaired. The repair, service and inspection rates are not supposed to be constant. At the end of a repair or of a service, the item is in a not too damaged state which can be chosen depending randomly on the state it is just leaving.

This model was introduced in [8] and studied by adding supplementary variables in order to obtain a Markov process. In this talk we will emphasize regenerative technics.

2. ASYMPTOTIC DISTRIBUTION

We want to compute the asymptotic distribution of the process $(\Phi_t)_{t\geq 0}$ defined as :

$$\pi(\eta) = \lim_{t\to+\infty} \mathbf{P}(\Phi_t = \eta).$$

In [2] and [3] semi-regenerative methods are used for such a study in a particular case : the distribution of the state of the item at the end of a repair (respectively a service) is the same for all the repairs (respectively services). In a first step, the semi-regenerative property of the times V_n when repairs or preventive maintenances are achieved is used. The Markov chain $(\Phi_{V_n})_{n\geq 0}$ is ergodic, therefore, if ν is its stationary distribution, we have :

$$\lim_{t\to+\infty} \mathbf{P}(\Phi_t = \eta) = \frac{1}{\sum_k \nu(k)\mathbf{E}(V_1 \,/\, \Phi_0 = k)} \sum_k \nu(k)\mathbf{E}\left(\int_0^{V_1} 1_{\{\phi_s=\eta\}}\, ds \,/\, \Phi_0 = k\right). \tag{1}$$

In a second step, all quantities involved in (1) are given in the form of matrix formulas. This is done by using that the inspection times and the beginning of repairs are semi-regenerative times.

In the general case, the same semi-regenerative technics can be used, but it is easier to work from the beginning with all the semi-regenerative times : inspection times, ends of repairs or services. In this case the stationary distribution of the embedded Markov chain is not given quite explicitly : it is the solution of a linear system of equations.

Formulas given in [8], [2] and [3] show that, in usual cases, the asymptotic distribution of Φ_t depends on the average duration of the repairs and preventive maintenances and not on the forms of their distributions. But it depends on the form of the inter-inspections distributions.

The semi-regenerative method allows to deal with general inter-inspections distributions, when the method with supplementary variables suppose the existence of rates, i.e. of density for these inter-inspections distributions.

3. AROUND MUT AND MTBF

Let $A(\infty)$ be the asymptotic availability of the item :

$$A(\infty) = \lim_{t\to+\infty} \mathbf{P}(\Phi_t \notin E_f) = \sum_{\eta\notin E_f} \pi(\eta).$$

Since the repair times $(U_n)_{n\geq 1}$ (i.e. the times where the repairs are ended) are semi-regenerative times for the process $(\Phi_t)_{t\geq 0}$, we have ([6] theorem 10.36) :

$$A(\infty) = \frac{MUT}{MTBF}, \tag{2}$$

where MUT and $MTBF$ are the Mean Up Time and the Mean Time Between Failure of the system, defined as :

$$MUT = \sum_i \tilde{\nu}(i)\mathbf{E}\left(\int_0^{U_1} 1_{\{\phi_s\in E_w\}}\, ds \,/\, \Phi_0 = i\right),$$

$$MTBF = \sum_i \tilde{\nu}(i)\mathbf{E}(U_1 \,/\, \Phi_0 = i),$$

$\tilde{\nu}$ being the stationary distribution of the Markov chain $(\Phi_{U_n})_{n\geq 0}$.

Let $N_f(t)$ be the number of failures of the item up to time t. In the reliability literature, the ROCOF (rate fo occurrence of failures) is defined as the derivative of $\mathbf{E}(N_f(t))$.

Using again that the repair times $(U_n)_{n\geq 1}$ are semi-regenerative times, and a martingale argument, we obtain :

$$ROCOF(\infty) = \lim_{t\to+\infty} ROCOF(t) = \frac{1}{MTBF}. \tag{3}$$

Remark : Introducing supplementary variables and using Shurenkov results ([10]), it is shown in [7] that formulas (2) and (3) are true for a large class of processes used in reliability theory, such as interacting semi-Markov processes.

It remains now to compute $ROCOF(\infty)$. We note that :

$$ROCOF(\infty) = \lim_{t\to+\infty} \frac{\mathbf{E}(N_f(t))}{t}.$$

In our model, we have $N_f(t) = \sum_{\eta\in E_f} N_\eta(t)$, where $N_\eta(t)$ is the number of visits to η before time t. For any $\eta \in E_f$, the entrance times to η are regenerative times for the process $(\Phi(t))_{t\geq 0}$. As a consequence, we obtain that :

$$\lim_{t\to+\infty} \frac{\mathbf{E}(N_\eta(t))}{t} = \frac{\pi(\eta)}{m(\eta)}, \tag{4}$$

where $m(\eta)$ is the mean sojourn time in η. Therefore :

$$ROCOF(\infty) = \sum_{\eta\in E_f} \frac{\pi(\eta)}{m(\eta)}.$$

At last, we obtain a way for calculating the $MTBF = 1/ROCOF(\infty)$. Using formula (2), we have also the MUT.

A way for proving (4) is the following : let η be a regenerative state for a general process $(\Phi_t)_{t\geq 0}$ and $\Psi(t)$ a kind of "additive functional" of this process (such as $N_\eta(t)$ or $\int_0^t 1_{\{\Phi_s=\xi\}}\, ds$). Wald lemma technic and the renewal theorem lead to :

$$\lim_{t\to\infty} \frac{\mathbf{E}(\Psi(t))}{t} = \frac{\mathbf{E}_\eta(\Psi(R_\eta))}{\mathbf{E}_\eta(R_\eta)},$$

where R_η is the first return time to η of the process $(\Phi_t)_{t\geq 0}$. We deduce that :

$$\lim_{t\to+\infty} \frac{1}{t}\, \mathbf{E}_\eta \left(\int_0^t 1_{\{\Phi_s=\eta\}}\, ds \right) = \frac{m(\eta)}{\mathbf{E}_\eta(R_\eta)},$$

and

$$\lim_{t\to+\infty} \frac{1}{t}\, \mathbf{E}_\eta(N_\eta(t)) = \frac{1}{\mathbf{E}_\eta(R_\eta)}.$$

Since

$$\lim_{t\to+\infty} \frac{1}{t}\, \mathbf{E}_\eta \left(\int_0^t 1_{\{\Phi_s=\eta\}}\, ds \right) = \pi(\eta),$$

we obtain (4).

4. COST FUNCTION AND OPTIMIZATION

In order to choose the best preventive maintenance policy, we can define a cost function and try to minimize it. It is natural to introduce costs as costs of inspections, costs of a service and a repair, hourly labor costs for services and repairs, ... The cost function to minimize is then the asymptotic average cost defined by

$$C = \lim_{t\to+\infty} \frac{\mathbf{E}(C(t))}{t},$$

$C(t)$ being the cost of using the item during the time interval $[0, t]$.

For estimating this cost, we have to compute quantities like :

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t 1_{\{\Phi_s=\eta\}}\, ds$$

which is equal to $\pi(\eta)$, and

$$\lim_{t \to +\infty} \frac{\mathbf{E}(N_\eta(t))}{t}$$

when η is a preventive maintenance or a repair state, i.e. a regenerative state, and this limit is given by (4).

So we have all the elements for computing the asymptotic cost.

The following step is to optimize the preventive maintenance by choosing the inter-inspections distributions. Numerical computations show that the optimum is obtained for deterministic inter-inspections durations.

In [2] and [3], Sophie Mercier gives the proof of this in a particular case.

5. RELIABILITY AND ASYMPTOTIC FAILURE RATE

Let $\lambda_\eta(\infty)$ be the asymptotic failure rate of a system when the initial state of the process is state η. In [5], it is shown that, for a large class of system, the convergence velocity of the system reliability is $e^{-\lambda_\eta(\infty)\,t}$ when starting from a regenerative state η and that $\lambda_\eta(\infty)$ satisfies the analog of a Cramer condition.

In [4] the authors show that, in the case of our preventive maintenance model, this asymptotic failure rate is the same for all semi-regenerative states η belonging to a same class of irreducibility. For this, they use [1], Theorem 2.6 of Chapter X. Moreover they are able to compute this asymptotic failure rate.

6. CONCLUSION

Our preventive maintenance model is a pleasant example where regenerative and semi-regenerative technics lead to effective calculations. By the way, some general results (formulas (2), (3), (4)) were given.
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