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Quasi-ergodic Nonstationary Queues

B.L. Granovsky* **and A.I. Zeifman****

Department of Mathematics, Technion-Israel Institute of Technology
** *Vologda State Pedagogical University*
and Vologda Scientific Coordinate Centre of CEMI RAS,
Vologda, Russia
email: zai@uni-vologda.ac.ru
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Evaluation of the rate of convergence of stochastic models, as time $t \to \infty$, has been a subject of investigation by generations of probabilists. During the last two decades a remarkable progress was made regarding time -homogeneous Markov chains, as a result of implementation and development of sophisticated techniques: coupling, logarithmic Sobolev inequalities, the Poincare inequality and its versions arising from the variational interpretation of eigenvalues, and duality. The above mentioned stream of nowadays research was motivated by new fields of applications, s.t. algorithms of Monte Carlo for simulation of Markov chains and enumeration algorithms in computers.

There is also a growing interest in time-nonhomogeneous Markov chains, see for instance [1], [7], [9]; such chains model a variety of queuing systems.

Our work is devoted to the estimation of the rate of convergence in different types of exponential convergence of nonhomogeneous queues. The main tool of our study is the method formulated by the second author in [10], [11] and subsequently developed and extended in a series of papers [4]-[6] written by the authors of the present paper. The method is based on tw logarithmic norm of a linear operator function and a special transformation of the reduced matrix of intensities of the considered Markov chain. In the present study we apply the method to a class of Markov queues with a special form of nonhomogenuity that is common in applications.

We want to mention that the initial motivation for the method considered came from B. V. Gnedenko [2], [3] and V. V. Kalashnikov [8].

The present research is the continuation of [4]-[6]. We deal with the class of nonhomogeneous birth and death processes (BDP) with intensities $\lambda_n(t) = \lambda_n a(t),\ \mu_n(t) = \mu_n b(t), \quad t \ge 0, \quad n \le N$ where $\mu_0 = 0,\ \lambda_n > 0, \quad n = 0, \dots, N-1, \quad \mu_n > 0,\ n = 1, \dots, N-1$. We suppose that there exist the limits

$$\lim_{n\to\infty} \lambda_n = \lambda > 0,\ \lim_{n\to\infty} \mu_n = \mu > 0, \tag{1}$$

if $N = \infty$. If $N < \infty$, we put $\lambda = \lambda_{N-1},\ \mu = \mu_N$.

The basic functions $a(t) \ge 0,\ b(t) \ge 0, \quad t \ge 0$ are supposed locally integrable on $[0, \infty)$ and asymptotically periodic. The last condition means that

$$a(t) = a_1(t) + a_2(t),\ b(t) = b_1(t) + b_2(t), \tag{2}$$

where

$$a_1(t + T_1) = a_1(t),\ b_1(t + T_2) = b_1(t), t \ge 0 \quad \lim_{t\to\infty} |a_2(t)| + |b_2(t)| = 0. \tag{3}$$

Put

$$a_m = \lim_{t\to\infty} \frac{1}{t} \int\limits_0^t a(u)\,du = \frac{1}{T_1} \int\limits_0^{T_1} a_1(u)\,du; \quad b_m = \lim_{t\to\infty} \frac{1}{t} \int\limits_0^t b(u)\,du = \frac{1}{T_2} \int\limits_0^{T_2} b_1(u)\,du. \tag{4}$$

We introduce the following **definition:** BDP is called **quasi-ergodic** if there exists a measure $\tilde{\mathbf{p}}$ on E, such that

$$\lim_{t\to\infty} \left\| \frac{1}{t} \int_0^t \mathbf{p}\,(\tau)\ d\tau - \tilde{\mathbf{p}} \right\| = 0$$

for any $\mathbf{p}\,(0) \in \Omega$, where Ω is the set of all stochastic vectors and $\| \bullet \|$ is l_1 -norm.

We study quasi-ergodicity and null-ergodicity for considered BDPs and its dependence on $a_m,\ b_m$.

We consider some homogeneous and nonhomogeneous queueing models and obtain some bounds of the rate of convergence.

Put

$$f_k = \frac{\Delta\lambda\mu_{k+1} - \lambda_{k-1}\mu + \sqrt{(\Delta\lambda\mu_{k+1} - \lambda_{k-1}\mu)^2 + 4\Delta\lambda\mu \cdot (\lambda_{k-1}\mu_{k+1} - \lambda_k\mu_k)}}{2\Delta\lambda\mu}, \tag{5}$$

where $\Delta > 0$, and λ, μ are defined in (1).

We denote $\tilde{E} = \{0, 1, \ldots, N-1\}$, if $N < \infty$ and $\tilde{E} = E$, if $N = \infty$.

Theorem 1. *Let for some* $\Delta > 1$

$$\inf_{k\in\tilde{E}} f_k = f > 0, \tag{6}$$

let c *be a positive number under conditions:*

$$c < 1;\ c < \frac{\mu_{k+1}}{\mu}, k \geq 0,\ c \leq f, \tag{7}$$

and let

$$\mu b_m - \Delta\lambda a_m > 0. \tag{8}$$

Then there exists a sequence $\mathbf{d} = (d_k, \quad k \in \tilde{E}) > \mathbf{0}$ *such that*

(i)

$$\underline{\alpha}\,(t) \geq l(t) = c \cdot (\mu b\,(t) - \Delta\lambda a\,(t))\,, \quad t \geq 0, \tag{9}$$

where $\liminf \frac{d_{k+1}}{d_k} > 1, \quad if \quad N = \infty.$

(ii) BDP $X(t), \quad t \geq 0$ *is weakly ergodic and*

$$\left\| \mathbf{p}^{(1)}\,(t) - \mathbf{p}^{(2)}\,(t) \right\| \leq \frac{4}{g} \cdot e^{-\int\limits_s^t l(u)du} \cdot \sum_{i\geq 1} q_i \left| p_i^{(1)}\,(s) - p_i^{(2)}\,(s) \right|, \quad 0 \leq s \leq t, \tag{10}$$

where $g = \inf_k d_k$, *and* $q_i = \sum\limits_{m=0}^{i-1} d_m$.

(iii) BDP $X(t), \quad t \geq 0$ *is quasi-ergodic.*

Put

$$h_k = \frac{\Delta\mu\lambda_{k-1} - \lambda\mu_k}{\Delta\lambda\mu}, \quad k \in \tilde{E}. \tag{11}$$

Theorem 2. *Let for some* $\Delta > 1$

$$\inf_{k \in \tilde{E}} h_k = h > 0, \tag{12}$$

let c be a positive number under conditions:

$$c < 1;\ c < \frac{\lambda_k}{\lambda}, k \geq 0,\ c \leq h, \tag{13}$$

and let

$$\lambda a_m - \Delta\mu b_m > 0. \tag{14}$$

Then there exists a sequence $\mathbf{d} = (d_k, \quad k \in \tilde{E}) > \mathbf{0}$ *such that*
(i)

$$\underline{\alpha}^{(0)}(t) \geq \theta(t) = c \cdot (\lambda a(t) - \Delta\mu b(t)), \tag{15}$$

where $\liminf \frac{d_{k+1}}{d_k} < 1, \quad if \quad N = \infty;$
(ii) BDP $X(t), \quad t \geq 0$ *is null-ergodic and*

$$\sum_{i=0}^{\infty} d_i p_i(t) \leq G \cdot e^{-\int_s^t \theta(u)du}, \quad 0 \leq s \leq t, \tag{16}$$

where $G = \sup d_k < \infty$.

The rest part of study is devoted to particular nonhomogeneous queues widely known in the literature: $M_t/M_t/S$ and $M_t/M_t/S/S$ queues and a multiserver queue with discouragements. We achieve an improvement of known estimates of the rate of convergence. We also obtain estimates of the expected length of queues.

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