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Queuing System with "Generalized Processor-sharing" and Dependence of Service Rate on Residual Work

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1. STATEMENT OF THE PROBLEM

Queuing system (QS) serves customers of a single class. If at a time t QS contains k customers, then in the interval $(t, t+dt)$, a customer will arrive at the system with probability $\lambda_k dt + o(dt)$, $dt \to 0$.

Customer arriving in QS begins to be served at once. The order of server occupation is the following. Customer arriving in a vacant QS occupies server number 1. When a new customer arrives during the service of the customer, it occupies server number 2 with the probability $\delta_2^{(2)}$ and server number 1 with the probability $\delta_2^{(1)}$. Then the customer served in server number 1 moves into server number 2. If there are k customers in the ith node, then they occupy the first k servers and the arriving customer with the probability $\delta_{k+1}^{(l)}$ occupies server number l, $l = 1, ..., k+1$. While the customers served in servers number $1, ..., l-1$ get their services, the customers served in servers number $l, l+1, ..., k$ begin to be served in servers number $l+1, ..., k+1$. If the service of the customer in server number l is completed, then the customers in servers number $1, ..., l-1$ remain there and the customers served in servers number $l+1, ..., k$ will be served in servers number $l, ..., k-1$.

The customer arriving in the QS in server j requires some work for this service. This work is a random value ξ with the distribution function $H(x)$, $H(0) = 0$, and the distribution density $h(x)$. If at time t there are k customers in the QS and $\xi_l(t)$ is the residual work from the moment t for completion of the customer service in the lth server, then the service rate in this server is equal to $\alpha_k \gamma_k^{(l)} \beta(\xi_l(t))$, $\alpha_k > 0$, $\gamma_k^{(l)} > 0$, $\beta(x)$, $\sum_{l=1}^{k} \gamma_k^{(l)} = 1$, that is $\frac{d\xi_l(t)}{dt} = -\alpha_k \gamma_k^{(l)} \beta(\xi_l(t))$.

Let us introduce the random process $\zeta(t) = \{\nu(t), \xi_1(t), ..., \xi_{\nu(t)}(t)\}$, where $\nu(t)$ is the number of customers in the QS at a time t.

In the framework of the problem statement $\zeta(t)$ is a piece-continuous Markov process [1]. Let us denote $F(k, x_1, ..., x_k) = \lim_{t\to\infty} \mathbb{P}\{\nu(t) = k, \xi_1(t) < x_1, ..., \xi_k(t) < x_k\}$, $P(0) = \lim_{t\to\infty} \mathbb{P}\{\nu(t) = 0\}$, $\bar{F}(k, x_1, ..., x_k) = \frac{\partial^k F(k, x_1, ..., x_k)}{\partial x_1 \cdots \partial x_k}$.

It is necessary to obtain the stationary state probabilities of this QS $P(k) = F(k, \infty, ..., \infty)$, $k = 0, 1, 2, ...$, the probability distribution density $\bar{F}(k, x_1, ..., x_k)$ and the distribution function $F(k, x_1, ..., x_k)$.

Remark. Generally speaking, the state of the system considered is the vector $\{\nu(t), \xi_1(t), ..., \xi_{\nu(t)}(t)\}$. But in the queuing theory the state of discrete components is understood as the system state, then $\{\nu(t), \xi_1(t), ..., \xi_{\nu(t)}(t)\}$ can be named the system microstate, taking into account that the state of the discrete components $\nu(t)$ is the system macrostate.

Later on we understand $\bar{F}(k, x_1, ..., x_k)$ and $F(k, x_1, ..., x_k)$ are non-normalizing probability distribution densities and probability distribution functions microstates. For their normalization it is necessary to divide them by $P(k)$. We understand $P(k)$ as the stationary distribution of macrostate probabilities. For simplification we will name them "states".

2. STATIONARY DISTRIBUTION OF MACROSTATE PROBABILITIES

Denote

$$(x_1, ..., x_k) = \vec{x}(k), (x_1, ..., x_{l-1}, x, x_l, ..., x_k) =$$
$$\mathcal{T}^+_{l,x}\vec{x}(k), (x_1, ..., x_{l-1}, x_{l+1}, ..., x_k) = \mathcal{T}^-_l \vec{x}(k).$$

The following theorems take place.

Theorem 2.1. *Let* $\int_0^\infty \frac{uh(u)}{\beta(u)}du < \infty$, $\beta(x) \equiv c$, $h(x) \neq \mu \exp^{-\mu x}$ *or* $\beta(x) \neq c$, *and a stationary ergodic distribution of the process* $\zeta(t)$ *exists, where* c *is a constant. In order to determine stationary distribution densities* $\bar{F}(k, \vec{x}(k))$ *and distribution functions* $F(k, \vec{x}(k))$ *of the QS in an evident analytical form in formulas*

$$\bar{F}(k, \vec{x}(k)) = P(0)\prod_{i=1}^{k}\frac{\lambda_{i-1}}{\alpha_i}\int_{x_i}^{\infty}\frac{h(u)}{\beta(u)}du, \tag{1}$$

$$F(k, \vec{x}(k)) = P(0)\prod_{i=1}^{k}\frac{\lambda_{i-1}}{\alpha_i}\left(x_i\int_{x_i}^{\infty}\frac{h(u)}{\beta(u)}du + \int_0^{x_i}\frac{uh(u)}{\beta(u)}du\right), k = 1, 2, ..., \tag{2}$$

where

$$P(0) = \left[1 + \sum_{k=1}^{\infty}\prod_{i=1}^{k}\frac{\lambda_{i-1}}{\alpha_i}\left(\int_0^{\infty}\frac{uh(u)}{\beta(u)}du\right)^k\right]^{-1}, \tag{3}$$

it is necessary and sufficient to have an execution of the following equalities:

$$1)\ \delta_k^{(l)} = \gamma_k^{(l)}, l = 1, ..., k, k = 1, 2, ... , \tag{4}$$

$$2)\ \int_0^{\infty}\frac{h(u)}{\beta(u)}du = \frac{1}{\beta(0)}. \tag{5}$$

Theorem 2.2. *For execution of the equality* $\int_0^\infty \frac{h(u)}{\beta(u)}du = \frac{1}{\beta(0)}$ *it is necessary and sufficient to have a determination of* $\beta(x)$ *in the formula*

$$\beta(x) = \beta(0)\frac{h(x)}{h_1(x)}, \tag{6}$$

where $h_1(x)$ *is some probability distribution density.*

Theorem 2.3 (invariance criterion). *Let* $\int_0^\infty \frac{uh(u)}{\beta(u)}du < \infty$, $\beta(x) \neq c$, *or* $h(x) \neq \mu \exp^{-\mu x}$, $\beta(x) \equiv c$, *and a stationary ergodic distribution of the process* $\zeta(t)$ *exist, where* c *is a constant. For the stationary state probabilities* $P(k)$ *not to depend on the functional form of the distribution* $H(x)$ *and* $\beta(x)$ *under a fixed* $\int_0^\infty \frac{uh(u)}{\beta(u)}du$ *which is an expectation of some random value and to be determined in an evident analytical form in formulas*

$$P(k) = P(0)\prod_{i=1}^{k}\frac{\lambda_{i-1}}{\alpha_i}\left(\int_0^{\infty}\frac{uh(u)}{\beta(u)}du\right)^k, k = 1, 2, ... , \tag{7}$$

where $P(0)$ *is determined by Theorem 1, it is necessary and sufficient to have an execution of equalities (4) and (5).*

Theorem 2.4. *For a stationary ergodic distribution of the process $\zeta(t)$ to exist it is sufficient to have an execution of the following conditions:*

$$1)\ \sum_{k=0}^{\infty}\frac{1}{\lambda_k}=\infty, \qquad 2)\ P(0)>0.$$

REFERENCES

1. Buslenko N.P., Kalashnikov V.V., Kovalenko I.N. *Lectures on complicated systems theory.* Moscow: Sov. radio, 1973 (in Russian).