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Optimization of Advertising Expenses in the Functioning of an Insurance Company

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The free capital of an insurance company can be allocated to draw new clients (by advertising). On the one hand, it intensifies the flow of money received by the company, and on the other hand, it increases the sum of insurance payments and diverts a sum of money for advertising. Therefore, the problem arises of studying the influence of advertising expenses on the characteristics of functioning of an insurance company, on an average capital in particular.

Let us assume that without advertising expenses, the functioning of an insurance company is described by the classical model [1]: the flow of insurance premiums is continuous, so that during the period Δt the capital of company increased by $C_0\Delta t$, and the insurance payments are independent random variables with the mean a. We also assume that the insurance claims epochs forms a Poisson flow with intensity λ_0.

Denote by $S(t)$ the capital of the company at time t. Assume that during the period $[t,\ t+\Delta t]$ the capital $u(t)S(t)\Delta t$ be spent to draw new clients, where $0 \le u(t) \le u_0 \le 1$. For $u(t) \ll 1$ we can assume that the money is inflow in proportion to the capital spent. However, the advertising expenses cannot give effect before some time τ, and have the property of an after-effect, i.e. after advertising expenses are made, they begin to operate in some time. Let us introduce the auxiliary function $R(t)$ connected with $S(t)$ by the relation

$$k\frac{dR}{dt} = -R(t) + u(t)S(t),$$

where the constant k defines the after-effect of the advertising and we shall assume that the advertising expenses lead to the increase of the intensity of the flow of insurance premiums from C_0 to $C_0 + C_1R(t-\tau)$. However, with increasing number of the company's clients, the number of insurance cases also increases. Therefore, the intensity of the insurance payments will increase from λ_0 to $\lambda_0 + \lambda_1 R(t-\tau)$. It is natural to assume that the increase of the number of clients does not change the premium loading, that is

$$\frac{C_0}{\lambda_0} = \frac{C_1}{\lambda_1}.$$

Thus, the process of the average capital $\overline{S}(t)$ variation is described by the equations

$$\frac{d\overline{S}(t)}{dt} = -u(t)\overline{S}(t) + (C_0 - \lambda_0 a) + (C_1 - \lambda_1 a)R(t-\tau),$$

$$k\frac{dR(t)}{dt} = -R(t) + u(t)\overline{S}(t),$$

with initial conditions $\overline{S}(0) = S_0$ and $R(t) = 0$ when $t \in [-\tau, 0]$.

The goal of the insurance company is to maximize the average capital $\overline{S}(\tau)$ at some time T by choosing an advertising strategy $u(t)$. The resulting optimization problem can be solved by means of Pontryagin's maximum principle for the systems being described by the equations with lags [2].

The Hamilton function for our problem has the form

$$H(t) = P_1(t)[-u(t)\overline{S}(t) + (C_0 - \lambda_0 a) + \gamma R(t-\tau)] +$$
$$+ \frac{1}{k} P_2(t)[-R(t) + u(t)\overline{S}(t)],$$

where $\gamma = C_1 - \lambda_1 a$ and conjugate variables $P_1(t)$ and $P_2(t)$ are defined in the segment $[T-\tau,\ T]$ by the system of equations

$$\frac{dP_1(t)}{dt} = -\frac{\partial H}{\partial \overline{S}} = u(t)\left[P_1(t) - \frac{1}{k}P_2(t)\right],$$

$$\frac{dP_2(t)}{dt} = -\frac{\partial H}{\partial R} = \frac{1}{k}P_2(t),$$

with boundary conditions $P_1(T) = 1$, $P_2(T) = 0$, and in the segment $[0,\ T-\tau]$ by the system of the equations

$$\frac{dP_1(t)}{dt} = -\frac{\partial H}{\partial \overline{S}} = u(t)\left[P_1(t) - \frac{1}{k}P_2(t)\right],$$

$$\frac{dP_2(t)}{dt} = -\frac{\partial H}{\partial R} - \left.\frac{\partial H}{\partial R(t-\tau)}\right|_{t=t+\tau} = \frac{1}{k}P_2(t) - \gamma P_1(t+\tau),$$

The control $u(t)$ maximizing the Hamilton function is

$$u(t) = \begin{cases} u_0 \text{ , if } \overline{S}(t)\left[\dfrac{1}{k}P_2(t) - P_1(t)\right] > 0, \\ 0 \quad \text{, if } \overline{S}(t)\left[\dfrac{1}{k}P_2(t) - P_1(t)\right] \le 0. \end{cases}$$

Hence the optimal control is relay in character, the switching points of the control are defined by the condition

$$\frac{1}{k}P_2(t) - P_1(t) = 0.$$

It can be shown that under condition $\gamma > 1$ (the condition of advertising efficiency) there is a unique switching point of the control (the moment of advertising expenses ceasing) defined by the condition

$$t^* = T - \tau - k\ln\frac{\gamma}{\gamma - 1}.$$

The optimal control is

$$u(t) = \begin{cases} u_0 \text{ , } t \le t^*, \\ 0 \quad \text{, } t \ge t^*. \end{cases}$$

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2. Yanushevskii R.T. *Control for Objects with Lags* [in Russian], Nauka, Moscow, 1978.