Информационные процессы, Том 2, № 2, 2002, стр. 209–210.

On the Asymptotics of Big Components in a Generalized Allocation Scheme

N. I. Kazimirov

Institute of Applied Mathematical Research,
Karelian Research Centre RAS
email: rishelie@karelia.ru
Received October 14, 2002

Let us consider non-negative integer random variables $\eta_1, \dots, \eta_N$ with a joint distribution defined by the sequence $b = (b_0, b_1, b_2, \dots)$ and the parameter n as follows. For all integer non-negative numbers $k_1, \dots, k_N$ which sum is equal to n we have

$$\mathsf{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = \frac{b_{k_1} \cdots b_{k_N}}{\sum\limits_{j_1+\dots+j_N=n} b_{j_1} \cdots b_{j_N}}. \tag{1}$$

In [1] it was shown that probabilities (1) can be represented as

$$\mathsf{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = \mathsf{P}\{\xi_1 = k_1, \dots, \xi_N = k_N |\, \xi_1 + \dots + \xi_N = n\}, \tag{2}$$

where $\xi_1, \xi_2, \dots$ are independent identically distributed random variables such that

$$\mathsf{P}\{\xi_1 = k\} = x^k b_k / B(x), \quad B(x) = \sum_k x^k b_k,$$

and the parameter x depends on N, n in such a way that simplifies obtaining of the corresponding local limit theorems for the sum of random variables $\xi_1, \dots, \xi_N$ (usually we determine x by the equation $\mathsf{E}\xi_1 = n/N$) [2].

If $\eta_1, \dots, \eta_N, \xi_1, \dots, \xi_N$ satisfy (2) then we say that these random variables form a *generalized allocation scheme* [1]. In books [1, 3, 4] one can find several examples of combinatorial tasks which can be reduced to a generalized allocation scheme. This allows us to resolve these tasks using asymptotical methods of the Probability theory when $N \to \infty$. Latest investigations [4, 5] show the value of such approach.

Let $\eta_{(1)} \le \dots \le \eta_{(N)}$ be a variational series of random variables $\eta_1, \dots, \eta_N$. If $p = o(N)$ as $N \to \infty$ then the component $\eta_{(N-p)}$ is called *big*. Recently interest in the research of limit distributions of big components of the generalized allocation scheme has been growing [6, 7].

Further we suppose that the support of b (supp$b$) has a maximum span of 1, $b_0 > 0$ and the radius of convergence of $\sum_k x^k b_k$ equals 1. By $\mathfrak{B}_N(\alpha)$ we denote a binomial random variable with parameters N and α. For a non-negative integer r denote $P_r = \sum_{k>r} x^k b_k / B(x)$.

Let m be a minimal natural number such that gcd(supp$(b_0, \dots, b_m)) = 1$. Below the symbols $C_1, C_2, \dots$ denote some positive constants.

Theorem 1. *Let $N, n \to \infty$ in such a way that $n/N \le C_1 < \sup_x \mathsf{E}\xi_1$. Let also the parameter x satisfy the equation $n/N = \mathsf{E}\xi_1$, and the following conditions be valid:*
(a) *$b_k \le C_2 b_l$ for all k and all $l < k$ such that $b_l > 0$; besides as $k \ge k_0$ we have $b_k > 0$;*
(b) *$N x^m \to \infty$; $0 < x \le C_3 < 1$;*

(c) $p^3 = o(N\mathsf{D}\xi_1)$;
(d) $C_4(p+1) \le NP_r \le C_5(p+1)$; $r \ge m$, $b_r > 0$;
(e) $C_6 \le b_k/b_{k+1} \le C_7$ *as* $k \ge k_0$.
Then
(A) *for* $n/N \ge C_8$ *and any fixed integer* h

$$\mathsf{P}\{\eta_{(N-p)} \le r+h\} = (1+o(1))\mathsf{P}\{\mathfrak{B}_N(P_{r+h}) \le p\},$$

(B) *for* $n = o(N)$ *we have*

$$\mathsf{P}\{\eta_{(N-p)} = r\} = (1+o(1))\mathsf{P}\{\mathfrak{B}_N(P_r) \le p\};$$
$$\mathsf{P}\{\eta_{(N-p)} = r^+\} = (1+o(1))(1-\mathsf{P}\{\mathfrak{B}_N(P_r) \le p\}),$$

where $r^+ = \min\{k|\ k > r, b_k > 0\}$.

From this theorem follow some known results about the asymptotics of distributions of the components $\eta_{(N-p)}$ of the Galton—Watson forest [5] for a fixed p. Besides this theorem gives us analogous results for $p \to \infty$ non-rapidly (see the condition (c)), and the asymptotics of big components of the random recursive forest for $n = O(N)$ [6].

Theorem 1 does not give us full description of the limit distribution of big components because from the condition $n/N \le C_1$ it follows that the parameter x does not approach the critical point — radius of convergence of $B(x)$.

Now there is an obvious need to create a common theory that would describe the behaviour of components of a generalized allocation scheme (2) in all zones of the range of parameters N, n with sufficiently weak conditions for the sequence b.

The research was supported by the Russian Foundation for Basic Research, grant 00–01–00233.

REFERENCES

1. Kolchin V. F. *Random Mappings.* New York: Springer, 1986.
2. Kazimirov N. I. One case of local limit theorem. *Proc. Inst. App. Math. Research KRC RAS,* Petrozavodsk, 2002, vol. 3.
3. Kolchin V. F., Sevastyanov B. A., Chistyakov V. P. *Random Allocations.* New York: Wiley, 1978.
4. Kolchin V. F. *Random Graphs.* Moscow: Fizmatlit, 2000.
5. Pavlov Yu. L. *Random Forests.* Utrecht: VSP, 2000.
6. Pavlov Yu. L., Loseva E. A. Limit distributions of maximum tree size in a random recursive forest. *Discr. Math. App.,* 2002 (in print).
7. Kazimirov N. I. On some conditions of absence of a giant component in a generalized allocation scheme. *Discr. Math. App.,* 2002 (in print).