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About the Linear-Quadratic Regulator Problem under a Fractional Brownian Perturbation — Complete Observation[1]

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Several contributions in the literature have been already devoted to the extension of the classical theory of continuous-time stochastic systems driven by Brownian motions to analogues in which the driving processes are fractional Brownian motions (fBm's for short). The tractability of the standard problems in prediction, parameter estimation and filtering is now rather well understood. Concerning optimal control problems, as far as we know, it is far from fully demonstrated (nevertheless, see [1] for a recent attempt in a general setting). Here our aim is to illustrate the actual solvability of control problems by exhibiting an explicit solution for the case of the simplest linear-quadratic model.

We deal with the fractional analogue of the so-called linear-quadratic Gaussian regulator problem in one dimension. The real-valued state process $X = (X_t,\, t \in [0,T])$ is governed by the stochastic differential equation

$$dX_t = a(t)X_t dt + b(t)u_t dt + c(t)dB_t^H\,,\ t \in [0,T]\,,\ X_0 = x\,, \tag{1}$$

which is as usual interpreted as an integral equation. Here x is a fixed initial condition, $B^H = (B_t^H,\, t \in [0,T])$ is a normalized fBm with the Hurst parameter H in $[1/2,1)$ and the coefficients $a = (a(t), t \in [0,T])$, $b = (b(t), t \in [0,T])$ and $c = (c(t), t \in [0,T])$ are fixed (deterministic) continuous functions. We suppose that X is completely observed and that a closed-loop control of the system is available in the sense that at each time $t \in [0,T]$ one may choose the input u_t in view of the passed observations $\{X_s, s \le t\}$ in order to drive the corresponding state, $X_t = X_t^u$ say. Then, given a cost function which evaluates the performance of the control actions, the classical problem of controlling the system dynamics on the time interval $[0,T]$ so as to minimize this cost occurs. Here we consider the quadratic payoff J defined for a control policy $u = (u_t, t \in [0,T])$ by

$$J(u) = \frac{1}{2}\mathbf{E}\big\{q_{T} X_{T}^2 + \int_0^T [q(t)X_t^2 + r(t)u_t^2]dt\big\}\,, \tag{2}$$

where q_T is a positive constant and $q = (q(t), t \in [0,T])$ and $r = (r(t), t \in [0,T])$ are fixed (deterministic) positive continuous functions.

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Our main goal here is to show that actually when the system (1) is driven by a fBm with some $H \in (1/2, 1)$ instead of a Brownian motion, an explicit solution to the optimal control problem under the performance criterion (2) is still available.

At first we derive a sufficient condition for optimality in $\mathcal{U}_H$. Given $u \in \mathcal{U}_H$ and $X = X^u$, we introduce the following backward stochastic differential equation in the pair of unknown $(\mathcal{F}_t^H)$-adapted processes $p = (p_t,\, t \in [0,T])$ and $\beta = (\beta_t,\, t \in [0,T])$:

$$dp_t = -a(t)p_t dt - q(t)X_t dt + \beta_t dM_t^H \,,\; t \in [0,T]\,;\; p_{_T} = q_{_T} X_{_T} \tag{3}$$

where M^H is a Gaussian martingale, called in [2] the *fundamental martingale associated to* B^H.

Lemma. *Suppose that* $u \in \mathcal{U}_H$ *is such that* $u = -(b/r)p$ *where* (p,β) *is a pair of* $(\mathcal{F}_t^H)$*-adapted processes which satisfies equation (3). Then* u *minimizes* J *over* $\mathcal{U}_H$.

In order to state our main result, for any fixed $s \in [0,T]$, we introduce the 2×2 matrix-valued function $\Gamma(.,s) = (\Gamma(t,s), t \in [s,T])$ where

$$\Gamma(t,s) = \begin{pmatrix} \pi(t) & \gamma(t,s) \\ \gamma(t,s) & \lambda(t,s) \end{pmatrix},$$

is the unique nonnegative symmetric solution of the backward Riccati equation

$$\dot\Gamma(.,s) = -a\big\{\mathcal{K}_H^c(.,s)e_1'\Gamma(.,s) + \Gamma(.,s)e_1[\mathcal{K}_H^c(.,s)]'\big\}$$

$$-q\mathcal{K}_H^c(.,s)[\mathcal{K}_H^c(.,s)]' + \frac{b^2}{r}\Gamma(.,s)e_1e_1'\Gamma(.,s)\,;\; \Gamma(T,s) = q_{_T}\mathcal{K}_H^c(T,s)[\mathcal{K}_H^c(T,s)]'\,,$$

with the vectors e_1 and $\mathcal{K}_H^c(t,s)$ in $\mathbf{R}^2$ given by

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \mathcal{K}_H^c(t,s) = \begin{pmatrix} 1 \\ K_H^c(t,s) \end{pmatrix}.$$

Here for $H \in (1/2,1)$ the function K_H^c is given by

$$K_H^c(t,s) = H(2H-1)\int_s^t c(r)r^{H-1/2}(r-s)^{H-3/2}\,dr\,,\; 0 \le s \le t\,,$$

$$\bar\gamma(s) = \gamma(s,s)\,; \quad \bar\lambda(s) = \lambda(s,s)\,,$$

and $k(t,s) = k_H^{\bar\gamma}(t,s)$:

$$\bar k(t,s) = -\kappa_H^{-1}s^{\frac{1}{2}-H}\frac{d}{ds}\int_s^t (r-s)^{\frac{1}{2}-H}\bar\gamma(r)dr.$$

Theorem. *There exists a unique opimal control* $\bar u$ *in* $\mathcal{U}_{ad}$ *and the optimal pair* $(\bar u, \bar X)$ *is governed on* $[0,T]$ *by the system*

$$\bar u_t = -\frac{b(t)}{r(t)}[\pi(t)\bar X_t + \bar v_t]\,; \quad \bar X_t = X_t^{\bar u}\,,$$

$$\bar v_t = \int_0^t [-a(s) + \frac{b^2(s)}{r(s)}]\bar v_s ds + \int_0^t [\frac{\bar k(t,s)}{c(s)} - \pi(s)]\{d\bar X_s - [a(s)\bar X_s + b(s)\bar u_s]ds\}$$

Moreover, the optimal cost is given by

$$J(\bar u) = \frac{1}{2}\{\pi(0)x^2 + \int_0^T \bar\lambda(t)d < M^H >_t\}\,.$$

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