Информационные процессы, Том 2, № 2, 2002, стр. 228–230.

KALASHNIKOV MEMORIAL SEMINAR

On Simulating Finite Markov Chains by the Splitting and Roulette Approach

V.B. Melas

St. Petersburg State University
email: viatcheslav.melas@pobox.spbu.ru

Received October 14, 2002

The splitting and Russian roulette approach, suggested by von Neumann [1], is one of possible ways to increase the efficiency of stochastic computer simulation of random processes. The idea of the approach consists in a parallel simulation of several independent paths of a random process corresponding to the behavior of a complex system number investigation. To each path procedures of splitting and roulette are applied. In most versions described in the literature (see, f.e., [2] or [3]) strong restrictions on transition probabilities are assumed. However, in [4] a more general technique was introduced and a corresponding mathematical theory was developed. The present paper is devoted to a brief review of the theory for the case of simulating finite Markov chains. We also suggest an illustrating example.

Let us consider a finite Markov chain with the state space $\Omega = \{0, 1, \ldots, m\}$, a transition matrix P and a stationary distribution $\pi = (\pi_0, \ldots, \pi_m)^T$. Assume that for an integer number N all elements of the matrix P^N are positive. The problem is to evaluate the vector π by simulation.

As the immediate simulation method we will consider the well known regenerative approach [5]. Let $\{i^*\}$ be the initial state. The method consists in the simulation of several paths of the chain beginning in $\{i^*\}$ till return to this state. As the estimator of π_i $(i = 0, 1, \ldots, m)$ the number of hits to the state $\{i\}$ by all paths divided by the number of all steps by all paths is used.

To describe the branching technique which is a formalization of the splitting and roulette approach let us introduce the notion of experimental design.

Let $\beta_0, \ldots, \beta_m$ be arbitrary fixed nonnegative numbers and it is possible that $\beta_i = 0$ if the state $\{i\}$ is a state in which every path will be cancelled.

When a path transits from a state i to a state j, $i \in \Omega$, $j \in \Omega \backslash \{i^*\}$ when $\beta_j \leq \beta_i$ we will simulate additionally η paths beginning in the state j where

$$\eta = \begin{cases} \lfloor \beta_j/\beta_i \rfloor - 1 \text{ with a probability } 1 - \alpha \\ \lfloor \beta_j/\beta_i \rfloor \quad \text{with a probability} \quad \alpha, \end{cases}$$

$\alpha = \beta_j/\beta_i - \lfloor \beta_i/\beta_i \rfloor$ and $\lfloor a \rfloor$ designates the integer part of a. When $\beta_j < \beta_i$ we will cancel the current path with the probability β_j/β_i. As the estimator of π_k $(k = 0, 1, \ldots, m)$ we will take the number of hits into the state k by all path multiplied by β_k and divided by the sum of all such numbers. One of β_k could be equal to 0 since $\pi_k = 1 - \sum_{i \neq k} \pi_i$ and one of the π_i could be recalculated by others.

We will call experimental design the discrete probability measure $\tau = \{\tau_0, \ldots, \tau_m\}$, $\tau_i \geq 0$ $(i = 0, \ldots, m)$, $\sum \tau_i = 1$, where

$$\tau_i = \beta_i \pi_i \Big/ \sum_{k=0}^{m} \beta_k \pi_k.$$

In [4] it was proved that the estimators are asymptotically unbiased and the covariance matrix of the estimators of $(\pi_1, \ldots, \pi_m)$ multiplied by all steps is approximately equal

$$D(\tau) = W^T B(\tau) W, \; W = (I - P_{\text{-}})^{-1},$$

where $P_{\_}$ is the matrix P with the first row and the first column rejected,

$$B(\tau) = \sum_{k=0}^{m-1} \frac{\pi_k^2}{\tau_k} B_k,\ B_k = (p_{ki}\delta_{ij} - p_{ki}p_{kj})_{i,j=1}^{m},$$

$$\delta_{ij} = 1 \text{ for } i = j \text{ and } \delta_{ij} = 0 \text{ for } i \neq j.$$

The immediate simulation method is a particular case when $\beta_0 = \beta_1 = \ldots = \beta_m$ that is $\tau = \pi$. The natural efficiency criterion for the problem is the determinant of $D(\tau)$. Since W does not depend on τ the criterion is reduced to $\det B(\tau)$.

A design τ^* will be called D-optimal design if it minimizes the magnitude of $\det B(\tau)$.

Consider a particular case of the Markov chain embedded into the random process, corresponding to the length of queue with one server and m places for waiting. We will consider the simplest case when the input stream is a simplest stream and the time of service is an exponentially distributed random value.

Let ρ be the load of the system. Then the matrix P is of the form

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & & \\ 1-\Delta & 0 & \Delta & 0 & & & \\ 0 & 1-\Delta & 0 & \Delta & & & \\ \vdots & & & & & & \\ & & & & & 1-\Delta & 0 & \Delta \\ & & & & & 0 & 1-\Delta & \Delta \end{pmatrix}, \Delta = \rho/(1+\rho),$$

$\pi_1 = \Delta\pi_0$, $\pi_{i+1} = \rho^i\Delta\pi_0$, $i = 1, \ldots, m-1$, $\pi_0 = 1/(1 + \Delta + \Delta\rho + \ldots + \Delta\rho^{m-1})$.

We can calculate by induction that

$$\det B(\tau) = \left(\prod_{i=1}^{m} \frac{\pi_i^2}{\tau_i}\right) \Delta^m (1-\Delta)^m.$$

From here we found that the D-optimal design is $\tau^* = \left\{0, \frac{1}{m}, \ldots, \frac{1}{m}\right\}$ and

$$I(\tau) = \left(\frac{\det B(\pi)}{\det B(\tau)}\right)^{1/m} = \left(\prod_{i=1}^{m} \pi_i\right)^{-1/m} \Big/ m$$

with $\tau = \tau^*$. Note that $I(\tau)$ is a natural measure of efficiency of the design τ. It means the ratio of the number of steps needed by immediate simulation for obtaining results with a given accuracy to the respective number for the splitting and roulette approach. In table 1 values of the efficiency are given.

Table 1. Values of I.

m	5	5	5	10	10	10
ρ	1/2	1/3	1/4	1/2	1/3	1/4
I	2.0	4.0	6.8	6.0	31.6	109

We can conclude that the technique is very powerful in this case. For practical implementation the optimal values $\beta_j = 1/\pi_j$, $j \neq 0$ could be evaluated through the current simulation results in the style of sequential approach. As the initial state we can take $\{i^*\} = \{1\}$.

We can offer to use the design τ^* for the more complicated chains, simulation whereas the efficiency could be evaluated numerically.

REFERENCES

1. Kahn, H. (1956). Use of different Monte Carlo sampling techniques // *Symposium on Monte Carlo methods*, ed. H. A. Meyer. New York: John Wiley and Sons, 146–190.
2. Ogibin, V.N. (1967). On applications of splitting and roulette in the calculation of particles transport by Monte Carlo technique // *Monte Carlo technique in radiation transport problems*. Moscow, 72–82 (in Russian).
3. Glasserman, P., P. Heidelberger, P. Shahabuddin, and T. Zajic. (1996). Splitting for rare simulation analysis of simple cases // *Proceedings of the 1996 Winter Simulation Conference*, IEEE: Computer Society Press, 302–308.
4. Ermakov, S. M., and V. B. Melas. (1995). *Design and analysis of simulation experiments*. Dordrecht, London: Kluwer Academic Publishers.
5. Crane, A. M., and D. L. Iglehart. (1974). Simulating stable stochastic systems. 2: Markov chains // *Journal of Association of Computing Machinery* 21 (1), 114–123.