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Sensitivity Analysis in Insurance and Finance[1]

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In any marketplace, regardless of its level of regulation, the ultimate sin is mispricing of goods and products. In insurance there is legislation in place laying down that prices be fair and reserves be adequate (the "principle of equivalence"). In finance the unwritten laws of the well-functioning market enforce that prices be consistent (the "principle of no arbitrage").

A distinct common feature of insurance and finance is that valuation is largely model-based. Therefore, correct pricing is very much a matter of properly specifying and calibrating mathematical models. In this perspective the sensitivity of prices to changes in the model assumptions is an issue of great importance.

In life insurance mathematics the sensitivity of premiums and reserves to changes in valuation elements was always an issue, and in modern mathematical finance the sensitivities of prices to changes in certain model parameters have attracted so much attention that special Greek symbols have been reserved for them. The total harvest of theoretical results – even from the two areas combined – remains a rather scanty fare, however, limited to simple text-book situations where explicit formulas exist. For more complex models aiming to mimic reality, one has to resort to numerical methods.

Another feature that insurance and finance have in common is that they deal with random phenomena developing over time. In such a setting any price (or reserve) is a conditional expected value given the past, and its functional form is the solution of some non-stochastic differential equation. Now the recipe: Upon differentiating this differential equation with respect to some model parameter, one obtains a differential equation for the sensitivity (derivative) of the price with respect to that parameter. Solving the two differential equations simultaneously, typically by some suitable numerical method, one determines the price and its sensitivity at any time between inception and termination of the contract. This device was proposed in a recent paper by Kalashnikov and Norberg (Scandinavian Actuarial Journal, 2002) dealing specifically with applications to life insurance. A forthcoming paper by Norberg (2002) applies the method to finance.

The idea is known and has been so for a while in certain other branches of dynamic system analysis. It has also been alluded to in the finance literature (e.g. Wilmott (1998), p. 113) but there merely as a side remark to the traditional calculation of 'Greeks' in simplistic models; its ramifications and potential are not yet widely perceived.

The method will be illustrated by two examples. The first is fetched from Kalashnikov and Norberg (2002) and deals with life insurance. Suppose that a person at age x purchases a T-year term insurance policy with sum insured 1 payable upon death and premium payable continuously at constant rate π contingent on survival. The reserve is defined, at any time during the contract period, as the conditional expected value of discounted future benefits less premiums, given survival of the insured. With interest rate r and mortality rate $\mu(y)$ at age y, the reserve at time $t \in [0, n]$ is given by

$$V(t,r) \;=\; \int_t^T e^{-\int_t^\tau (r+\mu(x+s)\,ds)}\, \{\mu(x+\tau) \;-\; \pi(r)\}\, d\tau \,, \tag{1}$$

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where the dependence on r has been visualized to allow of sensitivity analysis. The actuarial principle of equivalence requires balance at 0,

$$V(0,r) = 0\,, \tag{2}$$

which is a "budget constraint" on $\pi(r)$. Upon introducing the conditional expected values of discounted future benefits,

$$V_b(t,r) \;=\; \int_t^T e^{-\int_t^\tau (r+\mu(x+s))\,ds}\,\mu(x+\tau)\,d\tau\,,$$

and discounted "baseline contributions",

$$V_c(t,r) = \int_t^T e^{-\int_t^\tau (r+\mu(x+s))\,ds}\,d\tau\,,$$

(1) and (2) can be cast as

$$V(t,r) \;=\; V_b(t,r) \;-\; \pi(r)\,V_c(t,r)\,, \tag{3}$$

$$\pi(r) \;=\; \frac{V_b(0,r)}{V_c(0,r)}\,. \tag{4}$$

The functions $V_b(t,r)$ and $V_c(t,r)$ are easily computed by solving the (essentially ordinary) differential equations

$$\frac{\partial}{\partial t}V_b(t,r) = (r+\mu(x+t))\,V_b(t,r) \;-\; \mu(x+t)\,, \tag{5}$$

$$\frac{\partial}{\partial t}V_c(t,r) = (r+\mu(x+t))\,V_c(t,r) \;-\; 1\,, \tag{6}$$

subject to the natural conditions

$$V_b(T,r) \;=\; V_c(T,r) \;=\; 0\,. \tag{7}$$

Finally, $\pi(r)$ and the reserve function are computed by (3) and (4).

Let derivatives with respect to r by denoted by primes. The sensitivities $\pi'(r) = \frac{\partial}{\partial r}\pi(r)$ and $V'(t,r) = \frac{\partial}{\partial r}V(t,r)$ are now obtained by the device described above. Differentiating (5) – (7) gives

$$\frac{\partial}{\partial t}V_b{}'(t,r) = V_b(t,r) \;+\; (r+\mu(x+t))\,V_b{}'(t,r)\,, \tag{8}$$

$$\frac{\partial}{\partial t}V_c{}'(t,r) = V_c(t,r) \;+\; (r+\mu(x+t))\,V_c{}'(t,r)\,, \tag{9}$$

$$V_b{}'(T,r) \;=\; V_c{}'(T,r) \;=\; 0\,. \tag{10}$$

Finally, differentiating (3) and (4), gives

$$V'(t,r) \;=\; V_b{}'(t,r) \;-\; \pi'(r)\,V_c(t,r) \;-\; \pi(r)\,V_c{}'(t,r)\,, \tag{11}$$

$$\pi'(r) \;=\; \frac{V_c(0,r)\,V_b{}'(0,r) \;-\; V_c{}'(0,r)\,V_b(0,r)}{V_c(0,r)^2}\,, \tag{12}$$

from which the sensitivities are computed.

Sensitivity analysis has many potential uses in the context of insurance, notably in deliberations about placing technical elements on the safe side and in construction of simultaneous confidence bands for reserves and other functionals from statistical data.

The second example is from finance. Consider the celebrated Black-Scholes equation for the price $f(s,t)$ at time t and stock price s of a European call option with maturity T and strike K,

$$\frac{\partial}{\partial t} f(s,t) \;=\; f(s,t)\, r \;-\; \frac{\partial}{\partial s} f(s,t)\, rs \;-\; \frac{1}{2}\frac{\partial^2}{\partial s^2} f(s,t)\, \sigma^2\,, \tag{13}$$

subject to

$$f(s,T) = (s\,-\,K)_+\,. \tag{14}$$

By tradition, sensitivities are denoted by Greek letters and. For an example, look at

$$\rho = \frac{\partial}{\partial r} f(s,t)\,.$$

Differentiating (13) and (14) with repsect to r leads to

$$\frac{\partial}{\partial t}\rho(s,t) \;=\; \rho(s,t)\, r \;+\; f(s,t) \;-\; \frac{\partial}{\partial s}\rho(s,t)\, rs - \frac{\partial}{\partial s} f(s,t)\, s \;-\; \frac{1}{2}\frac{\partial^2}{\partial s^2}\rho(s,t)\, \sigma^2\,, \tag{15}$$

$$\rho(s,T) = 0\,. \tag{16}$$

Now, to determine f and ρ, solve the differential equations (13) and (15) subject to the conditions (14) and (16).

In the simple examples presented here there exist explicit expressions that can be discussed directly as functions of the parameters they involve. Still the proposed method is, presumably, the most diligent way of getting at the numbers.

The method extends straighforwardly to more complicated models and insurance products where explicit formulas are out of reach and only numerical results can be obtained. It applies to any conditional expected value one might be interested in, e.g. higher order moments and the probability distribution of discounted future net liabilities. It presents a host of problems for further inquiry, ranging from application to special models and products, via development of numerical methods, to theoretical studies of e.g. the existence of sensitivities and their derivatives with respect to time.

The paper will give a survey of existing results on sensitivity analysis in the mathematics of insurance and finance, address the issues listed above, and add new examples of applications that hopefully will sustain the relevance of the dynamic approach.

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