Информационные процессы, Том 2, № 2, 2002, стр. 246–251.

KALASHNIKOV MEMORIAL SEMINAR

On Multi-State Reliability Systems[1]

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Received October 14, 2002

To the memory of
V. Kalashnikov

1. INTRODUCTION

In complex systems with controllable reliability any complete system failure does not occur suddenly but usually is a result of accumulation of a sequence of many gradual failures. It stimulates consideration of systems with gradual failures of different types, or multi-state reliability systems (for an extensive recent bibliography, see please, [1]. In this paper we propose a general approach to describe, model and evaluate the most common reliability characteristics of complex hierarchical systems with various types of gradual failures. Such failures may change the state of the system and the quality of its operation, but do not necessarily lead to complete system failure. The reliability of the system is partially controllable and in the case of detection of some failure it is repaired. We deal with probabilistic aspects of modeling system reliability and focus on both of its common characteristics, steady state probabilities and reliability function. We omit structural properties of the system, because it requests the concrete model investigation.

2. A GENERAL MODEL

Consider some *complex hierarchical multi-component system* subjected to *gradual* (internal) *failures* of different types. Assume that the system is constructed from blocks and branches of several levels (see fig. 1). Each block and the following after branches and blocks forms a hierarchical subsystem of the same type as the main one. The blocks of the last (lowest) level will be referred as *units* and may be subjected to gradual failures of its own type. We will denote by L the maximal level of units, and it is not necessary that any unit belongs to this level. Units of different levels are possible. The reliability of each unit is partially controllable. In case of detection of a failure the unit is returned to its initial state (e.g. by replacing it with a new one of the same type). Some special combination F of units failure causes the whole system failure. Simple version of this model with full renewal of the system was considered in [3].

To specify the states space of the system and to define appropriate process describing its behavior introduce vector index $k = (i_1, i_2, \ldots, i_{L(k)})$ which determine each unit of the system as belonging to appropriate chain of blocks at any level with level of k-th unit denoted by $L(k)$. Denote also by $\mathcal{K}$ the set of these indices (and appropriate units) with number of units denoted by $K = \#(\mathcal{K})$. Then the *states space* of the system can be represented as $E = \{\mathbf{x} = (x_k : \ k \in \mathcal{K})\}$, where for any $k \in \mathcal{K}$ the integer x_k represents the state of the k-th unit in sense of its reliability. It can take different values, depending on its type, $x_k \in \{0, 1, \ldots n_k\}$, where the exhausted state of k-th unit is denoted by n_k. Notice that these numbers

[1] The paper was partially supported by the RFBR Grants No. 99-01-00034 and No. 01-07-90259.

have no specific physical sense but indicate only a possible level of gradual failure of the k-th unit. The value of $x_k = n_k$ means the full failure of k-th unit.

1	2		$n^{(1)}$
$(1,1)$	$(2,1)$		$(n^{(1)},1)$
$\vdots$	$\vdots$	$\cdots$	$\vdots$
$(1,n_1^{(2)})$	$(2,n_2^{(2)})$		$(n^{(1)},n^{(2)}_{n^{(1)}})$

Fig. 1. A complex hierarchical system.

To model the system functioning by a finite state Markov process we assume that the transition times from one gradual level to another, as well as the repair time of a failed unit have exponential distributions. The respective parameters may depend on the type of the unit $k \in \mathcal{K}$ and also on the entire system state $\mathbf{x}$. These general assumptions allow us to model the reliability of such a system by using the multi-dimensional Markov process

$$\mathbf{X} = \{X_k(t) : \ k \in \mathcal{K}, \ t \geq 0\},$$

with set of states E, which should be specified for any particular system.

To describe the system behavior let us introduce the following notations:

$$\begin{aligned}
\mathbf{e}_k &\quad \text{is a unit vector with } x_k = 1 \text{ and } x_i = 0 \text{ for } i \neq k;\\
\mathbf{x}_k(i) &\quad \text{is the vector } \mathbf{x} \text{ only } k\text{-th component of which is changed by } i;\\
\mathcal{J}(\mathbf{x}) &= \{k \in \mathcal{K} : \ x_k = 0\};\\
\mathcal{J}^c(\mathbf{x}) &= \{k \in \mathcal{K} : \ x_k \neq 0\};\\
\Gamma_k(\mathbf{x}) &= \{\mathbf{y} \in E : \ y_i = 0 \text{ for } \ i \in \mathcal{J}(\mathbf{x}\}, \ \text{where } k \text{ is the number of zeros components}\\
\Gamma^u &= \{\mathbf{x} : \ x_k = n_k \text{ at least for one } k \in \mathcal{K}\};
\end{aligned} \tag{1}$$

Additional assumption concerns the structure of transition intensities of such a process. The specific of the reliability models make it reasonable to suppose that this process can jump only in neighboring states in the case that gradual failure arise (i.e. a transition into a next level will occur), and fall into the state of appropriate hyper plane (with $x_k = 0$ if k-th unit is repaired). This means that the transition intensities have the following form

$$a(\mathbf{x},\mathbf{y}) = \begin{cases} \lambda_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \\ \mu_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x}_k(0), \end{cases} \tag{2}$$

with $\lambda_k(\mathbf{x}) = 0$ for k such that $x_k = n_k$ and $\mu_k(\mathbf{x}) = 0$ for $k \in \mathcal{J}(\mathbf{x})$. Put also for simplicity

$$\gamma_k(\mathbf{x}) = \lambda_k(\mathbf{x}) + \mu_k(\mathbf{x}) \qquad \text{and} \qquad \gamma(\mathbf{x}) = \sum_{k \in \mathcal{K}} \gamma_k(\mathbf{x}). \tag{3}$$

We will refer to the processes having these properties in possession as to *a Multi-State Reliability Process* (MSRP). The intensities $\lambda_k(\mathbf{x})$ and $\mu_k(\mathbf{x})$ will be called *failure* and *repair intensities* correspondingly.

Different constraints to the state space E and/or the failure set F of this process and various assumptions about dependence of transition intensities on the state give the possibility to model a number of particular cases. We begin with the simple unit model.

3. A SIMPLE UNIT MODEL SOLUTION

Consider the reliability model of some simple unit, subjected to gradual failures. These failures transfer the unit from one state to another, and these states are under control. It means that the failure can be found and repaired in such a way that after any repair the unit is returned back to the initial state. Suppose exponentially distributed time for any transition. Then the behavior of the unit is represented by a Markov process $X = \{X(t) : \ t \geq 0\}$ with discrete states space $E = \{0, 1, \ldots, n\}$ and transition intensities

$$a(x, y) = \begin{cases} \lambda(x) & \text{for } y = x + 1, \\ \mu(x) & \text{for } y = 0. \end{cases}$$

The Kolmogorov's system of differential equations for this process gets the form

$$\begin{aligned} &\frac{d\pi(0;t)}{dt} + \lambda(0)\pi(0) = \sum_{x \in E} \mu(x)\pi(x), \\ &\frac{d\pi(x;t)}{dt} + \gamma(x)\pi(x) = \lambda(x-1)\pi(x-1), \qquad x \in \{1, 2, \ldots, n\}. \end{aligned} \tag{4}$$

The system of complete balance equations for stationary probabilities of such a process can be obtained by passing to limit when $t \to \infty$, or eliminating the derivatives from the equations (4). It should be remarked that the detailed balance equations do not take place for this model, and a product form representation of steady state probabilities is not possible (see, for example, [2]). Nevertheless, these equations admit some general solution, represented in the following theorem.

Theorem 1. *The steady state probabilities of the MSRP for the simple unit with gradual failures have the form*

$$\pi(x) = g(x) \left[\sum_{0 \leq x \leq n} g(x) \right]^{-1} \quad \textit{with} \quad g(x) = \prod_{1 \leq i \leq x} \frac{\lambda(i-1)}{\gamma(i)} \quad \textit{and} \quad g(0) = 1. \tag{5}$$

The **Proof** can be obtained by the substitution of last expressions to the system of equations for stationary probabilities. □

From the theorem the simple consequence follows.

Corollary. *If failure set coincides with exhausted state $F = \{n\}$, the failure probability π_F of the single unit with gradual failures equals*

$$\pi_F = \pi_n = g(n) \left[\sum_{0 \leq x \leq n} g(x) \right]^{-1}. \quad \square$$

To calculate the reliability function of the simple unit with gradual controllable failures one should find the cumulative probability distribution function of the time to the first entrance in the state n for process $X(t)$. For this purpose one should solve the system of equations (4) for the process with initial state $X(0) = 0$ $(\pi(0;0) = 1)$ and absorbing state n. By setting $\mu_n = 0$ and denoting

$$\tilde{\gamma}(x;s) = \gamma(x) + s \qquad \text{and} \qquad \tilde{g}(x;s) = \prod_{1 \leq i \leq k} \frac{\lambda(x-1)}{\gamma(x;s)} \tag{6}$$

the solution of this system can be found in terms of Laplace transform.

Theorem 2. *The reliability function of the simple unit with gradual failures is*

$$R(t) = 1 - \pi(n;t), \tag{7}$$

where $\pi(n;t)$ *defined by its Laplace transform*

$$\tilde{\pi}(n;s) = \tilde{g}(n;s)\left[\tilde{\gamma}(0;s) - \sum_{1\le x\le n}\mu(x)\tilde{g}(x;s)\right]^{-1}. \tag{8}$$

The **Proof** follows from the solution of algebraic system of equations obtained by applying the Laplace transform to the system of equations (4). □

4. SOLUTION FOR A GENERAL MODEL

In terms of notations (1) the Kolmogorov's system of differential equations for the time dependent probabilities of the process states with transition intensities (2, 3) gets the form

$$\frac{d\pi(\mathbf{x};t)}{dt} + \gamma(\mathbf{x})\pi(\mathbf{x}) = \sum_{j\in\mathcal{J}^c(\mathbf{x})}\lambda_j(\mathbf{x}-\mathbf{e}_j)\pi(\mathbf{x}-\mathbf{e}_j) +$$

$$+ \sum_{j\in\mathcal{J}(\mathbf{x})}\sum_{1\le i\le n_j}\mu_j(\mathbf{x}_j(i))\pi(\mathbf{x}_j(i)), \qquad \mathbf{x}\in E. \tag{9}$$

The system of equations for stationary probabilities of the process can be obtained by eliminating derivatives (setting them equal to 0) in left side of (9), i.e. by taking the limit in both sides when $t\to\infty$. Its solution can be represented in algorithmic form. To do that denote by $p(\mathbf{x}) = p(\mathbf{0},\mathbf{x}) = p(\mathbf{0} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n = \mathbf{x})$ some path from the state $\mathbf{x}_0 = \mathbf{0}$ to the state $\mathbf{x}_n = \mathbf{x}$ through the hyper plane $\Gamma_k(\mathbf{x})$ to which the state $\mathbf{x}$ belongs, and by $\alpha = \alpha(\mathbf{x}_{i-1},\mathbf{x}_i)$ the coordinate (label of the unit), which should be changed in order to ensure the transition from the state $\mathbf{x}_{i-1}$ to the state $\mathbf{x}_i$. Denote also by $g(\mathbf{x}) = g(p(\mathbf{0},\mathbf{x}))$ the following function along this path

$$g(\mathbf{x}) = g(p(\mathbf{x})) = g(\mathbf{0} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n = \mathbf{x}) = \prod_{1\le i\le n}\frac{\lambda_{\alpha(\mathbf{x}_{i-1}\mathbf{x}_i)}(\mathbf{x}_{i-1})}{\gamma(\mathbf{x}_i)}, \tag{10}$$

and by $P_k(\mathbf{x})$ the set of all monotone paths from the state $\mathbf{0}$ to the state $\mathbf{x}$ through the hyper plane $\Gamma_k(\mathbf{x})$, containing this state. Denote now for any of hyper planes $\Gamma_k = \Gamma_k(i_1,\ldots,i_k) = \{\mathbf{x}:\ x_{i_1} = \cdots = x_{i_k} = 0\}$ the family of functions $G_k(\mathbf{x}),\ \ k = 0,1,\ldots,K$ recursivelly by the following way

$$A_k(\mathbf{x}) = \sum_{p\in P_k(\mathbf{x})} g(\mathbf{x}), \qquad \mathbf{x}\in\Gamma_k(\mathbf{x}); \tag{11}$$

$$B_k(\mathbf{x}) = \sum_{j\in\mathcal{J}_k(\mathbf{x})}\sum_{1\le i\le n_j}\frac{\mu_j(\mathbf{x}_j(i))}{\gamma(\mathbf{x})}G_{k-1}(\mathbf{x}_j(i)), \quad \mathbf{x}\in\Gamma_k(\mathbf{x}); \quad B_0(\mathbf{x})\equiv 0 \tag{12}$$

$$G_k(\mathbf{x}) = A_k(\mathbf{x}) + \sum_{\mathbf{0}<\mathbf{y}\le\mathbf{x},\,\mathbf{y}\in\Gamma_k(\mathbf{x})} g(\mathbf{y},\mathbf{x})B_k(\mathbf{y}), \quad \mathbf{x}\in\Gamma_k(\mathbf{x}); \tag{13}$$

Remark that due to the definition the functions $A_k(\mathbf{x})$ sutisfy to the recursive relations

$$A_k(\mathbf{x}) = \sum_{p\in P_k(\mathbf{x})} g(\mathbf{x}) = \sum_{j\in\mathcal{J}^c}\frac{\lambda_j(\mathbf{x}-\mathbf{e}_j)}{\gamma(\mathbf{x})}A_k(\mathbf{x}-\mathbf{e}_j). \tag{14}$$

Also the formula (13) can be rewritten recursivelly as folows

$$G_k(\mathbf{x}) = A_k(\mathbf{x}) + \sum_{\mathbf{y}\le\mathbf{x},\,\mathbf{y}\in\Gamma_k(\mathbf{x})} g(\mathbf{y},\mathbf{x})B_k(\mathbf{y}) =$$

$$= \sum_{j\in\mathcal{J}_k^c(\mathbf{x})} \frac{\lambda_j(\mathbf{x}-\mathbf{e}_j)}{\gamma(\mathbf{x})} \left[A_k(\mathbf{x}-\mathbf{e}_j) + \sum_{\mathbf{0}<\mathbf{y}\le\mathbf{x}-\mathbf{e}_j,\,\mathbf{y}\in\Gamma_k(\mathbf{x}))} g(\mathbf{y},\mathbf{x}-\mathbf{e}_j)B_k(\mathbf{y}) \right] + B_k(\mathbf{x}) =$$

$$= \sum_{j\in\mathcal{J}_k^c(\mathbf{x})} \frac{\lambda_j(\mathbf{x}-\mathbf{e}_j)}{\gamma(\mathbf{x})} G_k(\mathbf{x}-\mathbf{e}_j) + \sum_{j\in\mathcal{J}_k(\mathbf{x})} \sum_{1\le i\le n_j} \frac{\mu_j(\mathbf{x}_j(i))}{\gamma(\mathbf{x})} G_{k-1}(\mathbf{x}_j(i)). \tag{15}$$

In these notations the steady state probabilities of the considered system are given by the following

Theorem 3. *The functions $G_k(\mathbf{x})$ represent invariant measure of the MSRP and its steady state probabilities can be obtained by their normalization and have the form*

$$\pi(\mathbf{x}) = \left[\sum_{\mathbf{x}\in E} G(\mathbf{x}) \right]^{-1} \times G(\mathbf{x}) \quad \textit{for any} \;\; \mathbf{x}\in E. \tag{16}$$

Proof. The process is ergodic, thus the substitution of the formulas (11 – 13) to the system of equations for stationary probabilities prove the statment of the theorem. □

Corollary. *The failure probability of the system π_F equals to the sum of the probabilities of the states over all failure set,*

$$\pi_F = \sum_{x\in F} \pi(\mathbf{x}). \tag{17}$$

Especially, if the failure set of the system is reduced to only one of its units, then the failure function is $\pi_F = \pi_\Gamma$ □.

The reliability function of the system can be found as the distribution of the time to first entrance of the process $X(t)$ into the failure set F. This distribution is a solution of the system (9) with initial state $\pi(\mathbf{0};0)$, where any failure state $\mathbf{x}\in F$ is considered as an absorbing state. Consider here the case, when the set of failure states coincide with the set of all upper boundary states $F=\Gamma^u$, so that $\mu(\mathbf{x})=0$ for all $\mathbf{x}\in\Gamma^u$.

The use of Laplace transform simplifies the solition of system (9), and transforms it to corresponding algebraic system. The solution to the last system has the same structure as in Theorem 3. To represent this solution denote as before $\tilde\gamma(\mathbf{x};s) = s+\gamma(\mathbf{x})$, and change all notations (10–12) by analogous notations with additional argument "s", where also $\gamma(\mathbf{x})$ have to be changed by $\tilde\gamma(\mathbf{x};s)$.

Theorem 4. *The reliability function of the MSRP has the form*

$$R_S(t) = 1 - \pi_\Gamma(t), \tag{18}$$

where $\pi_\Gamma(t)$ is the distribution of the first entrance of $X(t)$ in the set Γ. It is given by

$$\pi_\Gamma(t) = \prod_{\mathbf{x}\in\Gamma} \pi(\mathbf{x};t),$$

and $\pi(\mathbf{x};t)$ have the Laplace transforms

$$\tilde\pi(\mathbf{x};s) = \left[\tilde\gamma(\mathbf{0};s)\tilde G(\mathbf{0};s) - \sum_{j\in\mathcal{K}} \sum_{1\le i\le j} \mu_j(i\mathbf{e}_j)\tilde G(i\mathbf{e}_j;s) \right]^{-1} \times \tilde G(\mathbf{x};s). \tag{19}$$

The **Proof** can be done by substitution of this expression into the algebraic system obtained by the applying of the Laplace transform to the system (9) with upper boundary states as absorbing ones and initial state at zero state. □

5. AN ALGORITHM

Recursive formulas (11 – 13) allow to propose an algorithm for calculation of the steady state probabilities and the reliability function in terms of Laplace transform of the time dependent probabilities. Bellow we represent Algorithm for the steady state probabilities calculation. Calculation of Laplace transform of the time dependent probabilities follows by the same procedure, but their inversion is a special problem.

In the Algorithm a numerical notation of states instead of vector ones is used. Thus all units k are numerated from 1 to K and all states x of the system from are numerated from $1 = \mathbf{0}$ to $N = \prod_{1 \le k \le K}(n_k + 1)$.

Algorithm.

Begin. Input necessary information about system.

Integers: $K,\ N$ Real: $\lambda_k(x),\ \mu_k(x),\ \ 1 \le k \le K,\ \ 1 \le x \le N$. Create Massives: $g(x),\ G_k(x)\ \ 1 \le k \le K,\ \ 1 \le x \le N$. Restrictions: Input some restrictions if any, for example, set of failure states F.

Step 1. Beginning from $k = 0$ calculate recursivelly $G_k(\mathbf{x})$ for all $k = 0, 1, 2, \ldots K$ and for all $\mathbf{x} \in E$ using formulas (13–12).

Step 2. Calculate steady state probabilities $\pi(\mathbf{x})$ accordingly formula (15), and failure probability π_F accordingly formula (17).

Step 3. Print results.

End.

Any additional performance characteristics of the system reliability can be calculated using steady state probabilities and some special structural properties of the system.

6. CONCLUSION

Only general solutions for the reliability of systems with gradual failures and appropriate algorithms are represented in this paper. Some special computer tools are needed for real system investigation. This is in the process.

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