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Numerical Analysis of Optimal Control Policies for Queueing Systems with Heterogeneous Servers[1]

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1. INTRODUCTION

The problem of optimal jobs assignment to heterogeneous servers arises in many applications. The problem of optimal jobs assignment for two heterogeneous servers with respect to minimization of long run average mean number jobs in the system was considered in [1], where it was shown that the policy, which minimizes the number of customers in the system, has a threshold property and consists in using the fastest server if necessary. For the multi-server system, these properties of an optimal policy were generalized in [2].

In the talk, an algorithm is proposed which gives the possibility to find optimal threshold levels for different values of system parameters and investigate their behavior. Some numerical examples are also included.

2. THE PROBLEM

Consider an $M/M/K/N-K$ $(K \le N < \infty)$ controllable queuing system with K heterogeneous exponential servers of intensities μ_k $(k = \overline{1,K})$, $N-K$ places in the buffer, and a Poisson input of jobs with the intensity λ. At the arrival times, the control consists in sending the arrived job to one of the idle servers, or to the queue (if it is not full). At the service completion times, the control consists in sending or not a job from the queue (if it is not empty) to some of the idle servers. Thus, any job is rejected only in the case if at the time of its arrival the buffer is full and all servers are busy. Being sent to some server a job cannot change it. The problem consists in minimizing the long run average mean number of jobs in the system.

3. THE MODEL

For modeling of the system operation, consider the controllable process $\{Z(t)\} = \{(X(t), U(t))\}$ with the observed process $\{X(t)\} = \{(Q(t), D(t))\}$ and controlling process $\{U(t)\}$. Here $Q(t)$ is the queue length at time t, and $D(t) = (D_1(t), \dots, D_K(t))$ describes the states of the servers at this time, $D_i(t) = 0$ or 1 depending on the idle or busy i-th server. The states space of the observed process is $E = \mathbf{N} \times \{0,1\}^K$ with the set $\mathbf{N} = \{0, 1, \dots, N-K\}$. The decision set A consists of $K+1$ elements, i.e. $A = \{0, 1, \dots, K\}$, and the decision "0" denotes not to occupy any server (send the arrived job to the queue), while the control "k" denotes to send the next job on the k-th

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server. Denote also by $L(t) = Q(t) + \sum_{1\le k\le K} D_k(t)$ the random process of the number of jobs in the system.

For each state $x = (q, d_1, \dots, d_K)$ denote by $d(x) = \sum_{1\le k\le K} d_k$ the number of busy servers, by $l(x) = q(x) + d(x)$ the number of jobs in the system, by $J_0(x)$ and $J_1(x)$ the sets of busy and idle servers respectively, $J_0(x) = \{j : d_j(x) = 0\}$, $J_1(x) = \{j : d_j(x) = 1\}$, and by $M_0(x) = \sum_{j\in J_0} \mu_j$, and $M_1(x) = \sum_{j\in J_0} \mu_j$ the total service intensities of the idle and busy servers.

Notice that the decision depends on the state of the observable component and the decision set $A(x)$ at the state x equals $A(x) = \{0\} \cup J_0(x)$.

Under the considered assumptions, the process $\{Z(t)\} = \{(X(t), U(t))\}$ is a Markov decision one with transition intensities

$$\lambda_{xy}(a) = \begin{cases} \lambda, & \text{for } y = x + e_a, \\ \mu_j, & \text{for } y = x - e_j + 1_{\{q(x>0\}}(-e_0 + e_a), \quad j \in J_1(x), \\ 0, & \text{otherwise,} \end{cases}$$

where $e_i = (0, \dots, 1, \dots, 0)$ denotes the $K + 1$-dimensional vector, the i-th component of which (beginning from 0-th) is one and all others are zeros.

As usual (see, for example, [3]) define a strategy δ, the probability distribution $\mathbf{P}_x^\delta$ of the process $\{Z(t)\}$ given the initial state x and strategy δ, the expectation $\mathbf{E}_x^\delta$ with respect to this probability distribution. Then in mathematical terms the problem can be represented as follows: minimize with respect to all admissible strategies the following expression

$$g(x;\delta) = \lim_{t\to\infty} \frac{1}{t} \mathbf{E}_x^\delta \int_0^t L(u)\,du.$$

4. THE OPTIMALITY EQUATION

As is well known (see, for example [3]) for a Markov decision problem with respect to the long run average minimization criterion an optimal strategy is a stationary Markov one, i.e. it is determined by the optimal policy $f = \{f(x) : x \in E\}$ which can be found from optimality equations for the process as a minimizer of its right hand. The optimality equations for the considered model can be obtained by usual methods and have a form (see [2])

$$\begin{aligned} v(x) = l(x) + \lambda \min_{k\in A(x)} v(x + e_k) + \sum_{j\in J_1(x)} \mu_j \,[1_{\{q(x)=0\}} v(x - e_j) + \\ + 1_{\{q(x)>0\}} \min_{k\in A(x-e_j-e_0)} v(x - e_j - e_0 + e_k)] - g, \end{aligned} \qquad (1)$$

where $g = \inf_\delta g(x, \delta)$ and $v = \{v(x) :, \ x \in E\}$ are the so called *gain* and *value functions* of the model, which are determined by these equations, being last one up to some constant.

From these equations, one can see that the function to be minimized, to which we will refer as a *Bellman function*, can be represented in the form

$$b(x,\, k) = v(x + e_k), \qquad k \in A(x). \qquad (2)$$

Based on the optimality equations (1) and the Bellman function of the model (2), the following results were proved in [2]:

Theorem. *For the system under consideration an optimal policy is of a threshold type, i.e. for each state x, there exists some level of the queue length $q^*(x)$ (depending on the collection of busy servers $J_1(x)$) such that it is necessary to occupy some server only if $q > q^*(x)$; in this case the fastest of idle servers should be occupied. On the other hand, if in some state x the optimal decision is to allocate a job to the queue then the same decision is optimal for all y with $q(y) \le q(x)$ and the same collection of busy servers $J_1(y) = J_1(x)$.*

5. AN ALGORITHM

The previous theorem describes only qualitative properties of the optimal policies. For the optimal threshold level calculation, an algorithm can be proposed. This algorithm is based on the Howard's iteration approach ([4]) and takes into account some specific properties of the problem. The algorithm consists of two general steps: Policy Evaluation, and Policy Improvement. To transform multidimensional massifs into a one-dimensional one, the following numeration of the states is used, where numbers are denoted by the same letter as the state,

$$\#(x) = q \cdot 2^K + \sum_{1 \le k \le K} 2^k d_{K+1-k} \equiv x, \quad x = \overline{0, I-1} \quad \text{with} \quad I = (N-K) \cdot 2^K$$

It is necessary to remind that only the relative value function $v = \{v(x) : \ x \in E\}$ can be found from these equations. It means that one of $v(x)$ has to be fixed as zero, for example. In the algorithm below we put $v(0) = 0$. The principal steps of the algorithm are shown below.

Policy Evaluation.

For a given policy $f_n = \{f_n(x) : \ x = \overline{0, I-1}\}$ solve the equations (1) with respect to g_n and $v_n = \{v_n(x) : \ x = \overline{0, I-1}\}$ with a given accuracy ϵ putting $v_n(0) = 0$.

Policy Improvement.

For a given solution g_n and $v_n = \{v_n(x) : \ x = \overline{0, I-1}\}$ find a new policy $f_{n+1} = \{f_{n+1}(x) : x = \overline{0, I-1}\}$, which minimizes the Bellman function of the model (2), with

$$f_{n+1}(x) = \operatorname{argmin}\{v_n(x + e_k) : \ k \in A(x)\}.$$

The Algorithm **stops** when two successive approximation of policies coincide.

6. EXAMPLES

Some numerical examples were calculated with the help of routine, followed by the algorithm. In these examples the system with $K = 3$ servers, buffer $N - K = 10$ is considered. The service intensities μ_k ($k = 1, 2, 3$), and the input intensity λ were varied. The results of the optimal policies calculations are summarized in the diagrams, shown in figures $1 - 3$, where changing of the thresholds levels for the second (pictures labeled by letter 'a') and third (pictures labeled by letter 'b') servers as functions of the first service intensity, μ_1, for different values of the second service intensity, $\mu_2 = \{0.1; 0.2; 0.3; 0.4; 0.5\}$, and different values of the input intensity $\lambda = \{1; 0.5; 0.1\}$ are shown. Corresponding curves are marked by the labels $\{1; 2; 3; 4; 5\}$.

The pictures show that the optimal policies really have a threshold property, and also they show that the optimal threshold levels of both servers monotonously increase with increasing service intensities of the first server, but the threshold levels of the third server decrease with the increase of the second service intensity. They show also an increase in the threshold levels of both servers when the input intensity decreases.

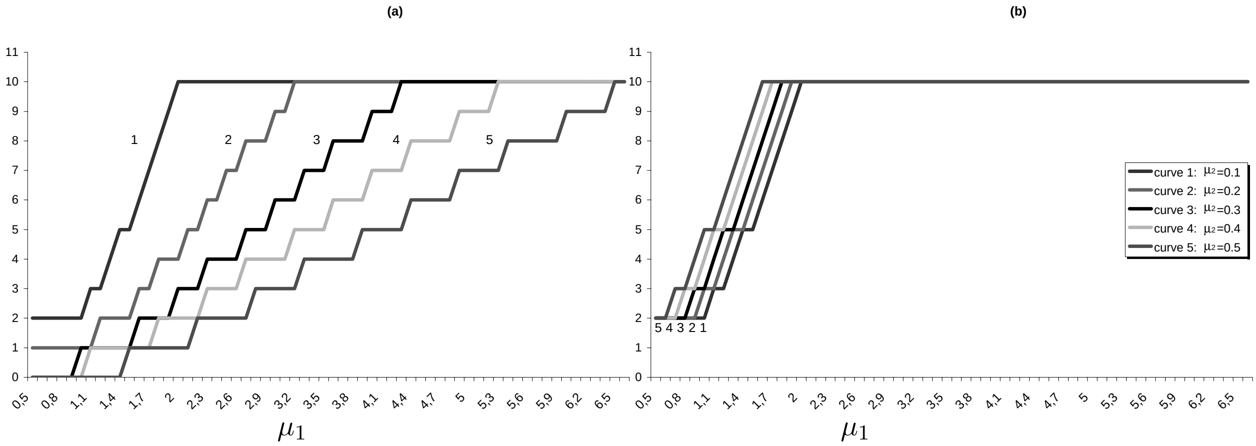

Fig.1. The threshold levels for second (a) and third (b) servers as functions of the first service intensity, μ_1, for different values of the second service intensity, μ_2, and the value of the input intensity $\lambda = 1$.

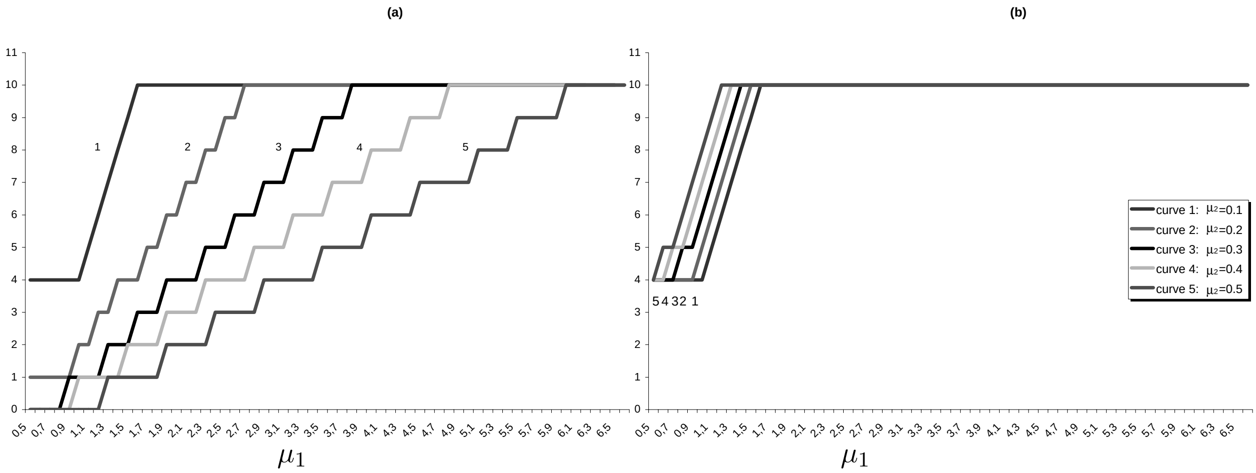

Fig.2. The threshold levels for second (a) and third (b) servers as functions of the first service intensity, μ_1, for different values of the second service intensity, μ_2, and the value of the input intensity $\lambda = 0.5$.

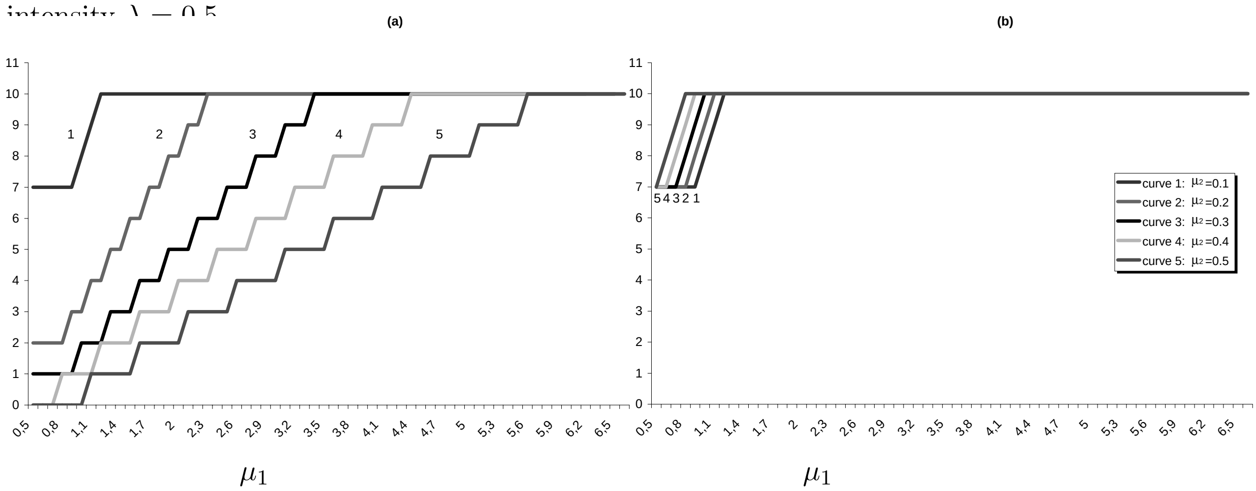

Fig.3. The threshold levels for second (a) and third (b) servers as functions of the first service intensity, μ_1, for different values of the second service intensity, μ_2, and the value of the input intensity $\lambda = 0.1$.

REFERENCES

1. W. Lin, P.R. Kumar, Optimal control of a queueing system with two heterogeneous servers. IEEE Trans. on Autom. Control 29 (1984) 696-703.
2. V.V. Rykov, Monotone Control of Queueing Systems with Heterogeneous Servers. *QUESTA*, 37 (2001), 391-403.
3. M. Yu. Kitaev, V.V. Rykov, *Controlled queueing systems* (CRC Press, New York, 1995).
4. R.A. Howard, *Dynamic programming and Markov processes* (Wiley, 1960).