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Asymptotics of Ruin Probabilities for Controlled Risk Processes in the Small Claim Case

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Let $S_t = \sum_{i=1}^{N_t} Y_i$ be the aggregate claims process, where $\{N_t\}$ is a Poisson process with rate λ. The claim sizes $\{Y_i\}$ are iid, strictly positive and independent of the claim arrival process. We denote by Y a generic random variable, by $M_Y(r) = \mathbb{E}[\exp\{rY\}]$ its moment generating function and by $G(y)$ its distribution function. The insurer follows a strategy $(A(u), b(u))$ of feedback form, where $(A(u), b(u)) \in \mathcal{A} \subset [0, \infty) \times [0, 1]$. The following cases have been investigated in [1, 2, 3]:

$$
\begin{aligned}
&\mathcal{A} = [0, \infty) \times \{1\}, && \text{no reinsurance,}\\
&\mathcal{A} = \{0\} \times [0, 1], && \text{no investment,}\\
&\mathcal{A} = [0, \infty) \times [0, 1], && \text{investment and reinsurance}
\end{aligned}
$$

where $A(u)$ denotes the amount invested into a risky asset, modelled as a geometric Brownian motion

$$\mathrm{d}Z_t = \mu Z_t \,\mathrm{d}t + \sigma Z_t \,\mathrm{d}W_t \,,$$

$\{W_t\}$ is a standard Brownian motion independent of $\{S_t\}$ and $b(u)$ is the retention level in proportional reinsurance, i.e. if a claim Y occurs at the time where the surplus is u (before the claim payment) then the insurer pays $b(u)Y$ and the reinsurer pays $(1 - b(u))Y$. For this reinsurance cover the insurer has to pay a continuous premium at rate $c(b(u))$. As in [3] we assume that $c(b)$ is strictly decreasing, $c(1) = 0$, and that $c < c(0) < \infty$, where c is the rate at which the insurer receiver premiums.

Under the chosen strategy the surplus process X is given by

$$\mathrm{d}X_t = (c - c(b(X_t)) + \mu A(X_t))\,\mathrm{d}t + \sigma A(X_t)\,\mathrm{d}W_t - b(X_{t-})\,\mathrm{d}S_t \,, \qquad X_0 = u \,.$$

The time of ruin is $\tau^{A,b} = \inf\{t \geq 0 : X_t < 0\}$, and the ruin probability is $\psi^{A,b}(u) = \mathbb{P}[\tau^{A,b} < \infty]$. The control function is $\psi(u) = \inf_{\mathcal{A}} \psi^{A,b}(u)$. In order that $\psi(u) < 1$ we have to assume that $c > \lambda \mathbb{E}[Y]$ in the case without investment. If investment is possible the positive safety loading can be achieved by investment.

As in [1, 2, 3] we suppose that $\psi(u)$ is twice continuously differentiable. Then $\psi(u)$ solves the Hamilton-Jacobi-Bellman equation

$$\inf_{(A,b)\in\mathcal{A}} \tfrac{1}{2}\sigma^2 A^2 \psi''(u) + (c - c(b) + \mu A)\psi'(u) + \lambda(\mathbb{E}[\psi(u - bY)] - \psi(u)) = 0,$$

where we let $\psi(u) = 1$ for $u < 0$. The optimal strategy $(A(u), b(u))$ are the values of A, b in the Hamilton-Jacobi-Bellman equation for which the infimum is taken.

Let $R(A, b)$ be the solution to

$$\lambda(M_Y(br) - 1) - (c - c(b) + \mu A)r + \tfrac{1}{2}\sigma^2 A^2 r^2 = 0 \,,$$

and $R = \sup_{(A,b)\in\mathcal{A}} R(A,b)$. Let (A^*, b^*) denote the parameters at which the supremum is attained.

We first consider the small claim case. The process

$$M_t = \exp\Big\{-R(X_{\tau\wedge t} - u) - \int_0^{\tau\wedge t} \theta(X_s)\,\mathrm{d}s\Big\}\,,$$

is a martingale where

$$\theta(u) = \lambda(M_Y(b(u)R) - 1) - (c - c(b(u)) + \mu A(u))R + \tfrac{1}{2}\sigma^2 A^2(u) R^2\,.$$

Define the measure $\mathbb{P}^*[A] = \mathbb{E}[M_t; A]$ on $\mathcal{F}_t$. Then $\mathbb{P}^*[\tau < \infty] = 1$ and

$$\psi(u) = \mathbb{E}^*\Big[\exp\Big\{RX_\tau + \int_0^\tau \theta(X_s)\Big\}\Big]\,\mathrm{e}^{-Ru}\,..$$

Upper and lower Lundberg bounds can be obtained. Let $\zeta = \limsup \psi(u)\mathrm{e}^{Ru}$. Then we show that for any $\varepsilon, n > 0$ there is an interval $[u_0 - n, u_0]$ on which $|\psi(u)\mathrm{e}^{Ru} - \zeta| < \varepsilon$. Using this the change of measure formula yields convergence of $\psi(u)\mathrm{e}^{-Ru}$. Further considerations then imply that $A(u) \to A^*$ and $b(u) \to b^*$ as $u \to \infty$.

Suppose now $M_Y(r) = \infty$ for all $r > 0$ and that investment is possible. Then we show that $\psi(u)\mathrm{e}^{Ru}$ converges, possibly to zero. Moreover, $\psi(u)\mathrm{e}^{(R+\varepsilon)u} = \infty$ for all $\varepsilon > 0$. The strategy also converges, $b(0) \to b^* = 0$ and $A(u) \to A^* < \infty$ as $u \to \infty$. Moreover, if $\liminf_{b\to 0} b^{-1}(c(0) - c(b)) > \lambda\mathbb{E}[Y]$, we find that $b(u) > 0$ for all u.

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