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Reversible Markov Processes on General Spaces: Spatial Birth-Death and Queueing Processes

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The first part of this study characterizes several properties of a reversible Markov jump process $X \equiv \{X_t : t \geq 0\}$ on a general measurable space $(E, \mathcal{E})$ with transition rate kernel $q(x, A)$. The process (or the kernel q) is reversible *reversible* with respect to π, if π is a measure on $\mathbb{E}$ such that

$$\pi(dx)q(x\,dy) = \pi(dy)q(y\,dx).$$

Reversibility was introduced by Kolmogorov; see the review [1] and its applications to queueing in [3, 6, 8], which are for processes on countable state spaces. We present a canonical representation of the stationary distribution of X on a general state space. This involves representing two-way communication by certain Radon-Nikodym derivatives for measures on product spaces, using a result from [7]. This is not needed for classical processes on discrete spaces or for kernels with with density functions (e.g., $q(x\,dy) = r(x, y)\mu(dx)$). Included is a Kolmogorov criterion that establishes the reversibility of ψ-irreducible Markov jump processes [5].

The second part of the study derives stationary distributions for two classes of reversible measure-valued Markov processes:
(1) Spatial birth-death processes with single and multiple births and deaths (the total population is never infinite, which is different from infinite-population systems [2, 4]).
(2) Spatial queueing systems in which customers move in a space where they receive services, analogous to services in queueing networks [3, 6, 8].
Sufficient conditions for ergodicity of spatial queues are also presented.

The following is the main result for a *spatial birth-death process*. Consider a system in which discrete units enter a measurable space $(\mathbb{E}, \mathcal{E})$ for processing and then leave the space. We represent the state of the system over time by a Markov jump process $X = \{X_t : t \geq 0\}$ with state space $(\mathbb{N}, \mathcal{N})$, the space of all finite counting measures ν on $\mathbb{E}$. That is, X is a measurable map from a probability space to $(\mathbb{N}, \mathcal{N})$, and $X_t(A)$ is the random number of units in $A \in \mathcal{E}$ at time t. Whenever the process X is in a state ν, the time to the next potential arrival (birth) from the outside into the set $A \in \mathcal{N}$ is exponentially distributed with birth-rate kernel $\lambda(\nu, A)$, where $\sup_{\nu \in \mathbb{N}} \lambda(\nu, \mathbb{E}) < \infty$. Also, for each unit located at $x \in \mathbb{E}$, the time to its departure (death) is exponentially distributed with death rate $\gamma(\nu, x)$, which is positive when $\nu(dx) > 0$. Then X is a Markov jump process with transition rate kernel

$$q(\nu, C) = \int_{\mathbb{E}} \lambda(\nu, dx) 1(\nu + \delta_x \in C) + \int_{\mathbb{E}} \nu(dx)\gamma(\nu, x) 1(\nu - \delta_x \in C), \quad \nu \in \mathbb{N},\ C \in \mathcal{N}. \tag{1}$$

The following result characterizes the reversibility of X in terms of the measures H_n on $\mathbb{E}^n$ defined by

$$H_n(dx_1 \cdots dx_n) \equiv \frac{1}{n!} \prod_{k=1}^{n} \gamma(\nu_k, x_k)^{-1} \lambda(\nu_{k-1}, dx_k), \tag{2}$$

where $\nu_0 = 0$ and $\nu_k \equiv \sum_{i=1}^{k} \delta_{x_i}$.

Theorem 1. *The Markov process X with transition rate kernel (1) is reversible with respect to a measure with an atom at 0 if and only if*

$$H_n(dx_1 \cdots dx_n) = H_n(dx'_1 \cdots dx'_n), \tag{3}$$

for any permutation $x'_1 \cdots x'_n$ of $x_1 \cdots x_n$ and $n \geq 1$. In this case, X is reversible with respect to

$$\pi(C) = \sum_{n=0}^{\infty} \int_{\mathbb{E}^n} H_n(dx_1 \cdots dx_n) 1(\sum_{i=1}^{n} \delta_{x_i} \in C), \quad C \in \mathcal{N}, \tag{4}$$

where the term in the sum for $n = 0$ is $1(0 \in C)$.

We now describe a *spatial queueing system* in which a measure-valued Markov process $X \equiv \{X_t : t \geq 0\}$ as above represents the locations of units (customers) that move in the space $\mathbb{E}$ where they are processed. The X jumps from the state ν to the state

$$T_{xy}\nu \equiv \nu - \delta_x + \delta_y \in \mathbb{N},$$

when a unit at x moves to the location y. Here $\delta_0 = 0$ when $x = 0$, and $\nu(\{x\}) > 0$ when $x \in \mathbb{E}$. We assume that the transition rate kernel for the process is

$$q(\nu, C) = \int_{\mathbb{E}} r(\nu, T_{0y}\nu) \lambda(0, dy) 1(T_{0y}\nu \in C) \tag{5}$$

$$+ \int_{\mathbb{E} \times \overline{\mathbb{E}}} r(\nu, T_{xy}\nu) \lambda(x, dy) 1(T_{xy}\nu \in C), \quad \nu \in \mathbb{N},\ C \in \mathcal{N}. \tag{6}$$

Here $r(\nu, T_{xy}\nu)$ is a *departure-attraction* rate at which a unit tends to depart from x and be attracted to y. This is analogous to a service and attraction rate in queueing networks. The $\lambda(x, dy)$ is a Markovian routing kernel as above with stationary distribution α. The multiplication $r(\nu, T_{xy}\nu)\lambda(x, dy)$ or compounding is similar to the compounding of service and routing rates in Jackson networks.

We assume that λ and $r(\nu, T_{xy}\nu)$ are such that $q(\nu, \mathbb{N})$ is finite and the process is regular. These conditions are true if

$$\sup_{x \in \overline{\mathbb{E}}} \int_{\overline{\mathbb{E}}} r(\nu, T_{xy}\nu) \lambda(x, dy) < \infty.$$

As invariant measure for the process X is generally not tractable, but it is tractable when X is reversible. For this we assume that routing kernel $\lambda(x, A)$ on $\mathbb{E}$ is reversible with respect to α.

Theorem 2. *If $f(\nu)r(\nu, \eta)$ is symmetric for some positive function f on $\mathbb{N}$, then X is reversible with respect to $\pi(d\nu) = f(\nu)\pi_\alpha(d\nu)$, where π_w is the Poisson distribution with intensity α:*

$$\pi_\alpha(C) = e^{-\alpha(\mathbb{E})}[1(0 \in C) + \sum_{n=1}^{\infty} \int_{\mathbb{E}^n} \frac{1}{n!} \alpha(dx_1) \cdots \alpha(dx_n) 1(\sum_{k=1}^{n} \delta_{x_k} \in C)], \quad C \in \mathcal{N}.$$

Example 1. Consider the process with departure-attraction rate

$$r(\nu, T_{xy}\nu) = \frac{u(\nu - \delta_x)}{u(\nu)} \frac{v(\nu + \delta_y)}{v(\nu)},$$

where u and v are positive functions. The first ratio is the rate at which a unit at x departs from its location, and the second ratio is the rate at which a unit is attracted to the location y. In this case, $u(\nu)v(\nu)r(\nu, \eta)$ is symmetric, and hence X is reversible with respect to $\pi(d\nu) = u(\nu)v(\nu)\pi_\alpha(d\nu)$.

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