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KALASHNIKOV MEMORIAL SEMINAR

On the Identifiability of a Two-Factor Model with Correlated Residuals

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Consider the following two-factor model:

$$X_j^1 = \theta_j^1 H_1 + Y_j^1, \quad j = 2, ..., n_1,$$
$$X_j^2 = \theta_j^2 H_2 + Y_j^2, \quad j = 2, ..., n_2, \tag{1}$$

where $X = (X_1^1, ..., X_{n_1}^1, X_1^2, ..., X_{n_2}^2)$ is a vector of observable variables with a vector of means m_X and covariance matrix Σ_X; $\theta_X = (\theta_1^1, ..., \theta_{n_1}^1, \theta_1^2, ..., \theta_{n_2}^2)$ is a vector of parameters; $H = (H_1, H_2)$ is a vector of latent variables normally distributed with $E(H_1) = E(H_2) = 0$ and $Var(H_1) = Var(H_2) = 1$; $Y = (Y_1^1, ..., Y_{n_1}^1, Y_1^2, ..., Y_{n_2}^2)$ is a vector of residuals normally distributed with positive definite covariance matrix; Y and H are independent.

We will assume that probabilistic relationships among components of the vector Y are represented by a Bayesian network [1] with a structure S_Y, that is a directed acyclic graph with nodes identified with components of the vector Y, and a vector of parameters θ_Y. The model (1) is a generalization of the single-factor model with correlated residuals suggested in [2].

The model (1) is said to be identifiable if the vector of parameters $\theta = (\theta_X, \theta_Y)$ can be uniquely determined by m_X and Σ_X.

Transform the structure S_Y by the following way. Connect by edges all disconnected by arcs nodes and then delete all arcs. Obtained graph $\overline{S}_Y$ is called complementary graph of the structure S_Y.

Let C_1 be a cycle that consists of nodes $Y_1, ..., Y_k$ and C_2 be a graph that consists of a cycle with nodes $Y_1, ..., Y_{i_1}$, a simple chain with nodes $Y_{i_1}, ..., Y_{i_2}$ and a cycle with nodes $Y_{i_2}, ..., Y_n$.

Define two functions:

$$F_1(C_1) = \sum_{i=1}^{k-1} \chi(Y_i, Y_{i+1})(-1)^{i+1} + \chi(Y_1, Y_k)(-1)^{k+1}$$

and

$$F_2(C_2) = \sum_{j=1}^{i_1-1} \chi(Y_j, Y_{j+1})(-1)^{i+1} + \chi(Y_1, Y_{i_1})(-1)^{i_1+1}+$$
$$2(\sum_{j=i_1}^{i_2-1} \chi(Y_j, Y_{j+1})(-1)^{j+1})+$$
$$\sum_{j=i_2}^{n-1} \chi(Y_j, Y_{j+1})(-1)^{j+1} + \chi(Y_n, Y_{i_2})(-1)^{i_2},$$

where $\chi(Y_i, Y_j) = 0$, if $Y_i, Y_j \in \{Y_1^1, ..., Y_{n_1}^1\}$ or $Y_i, Y_j \in \{Y_1^2, ..., Y_{n_2}^2\}$ and $\chi(Y_i, Y_j) = 1$ in opposite case.

The followig theorem provides identifiability conditions for model (1).

Theorem. *Assume that* Σ_X^{-1} *does not contain zero elements and* $\theta_1^1 > 0$, $\theta_1^2 > 0$. *A necessary and sufficient condition for model* (1) *to be identified is that every connectivity component of the graph* $\overline{S}_Y$ *contains at least one odd cycle and at least one connectivity component contains either even cycle* C_1 *for that* $F_1(C_1) \neq 0$ *or subgraph* C_2 *that consists of two odd cycles connected by simple chain for that* $F_2(C_2) \neq 0$.

REFERENCES

1. *Heckerman D.* A tutorial on learning Bayesian networks. Tech. Report MSR-TR-95-06, Microsoft, WA, 1995.
2. *Vicard P.* On the identification of a single-factor model with correlated residuals // Biometrika, 2000, v. 87, p. 199-205.