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Quality Properties of Risk Models Under Stochastic Interest Force[1]

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1. INTRODUCTION

In [1] a risk model under a stochastic interest force was constructed and some results concerning the behavior of the ruin probability in this model were obtained. However, the properties of this model (even qualitative) were not investigated.

In this paper the problem of qualitative investigation of this model is considered and main formulas characterizing its dynamics are obtained. This paper develops classical results on the behavior of the ruin probability under conditions of permanent inflation and constant interest force and gives a generalization of a Lindley chain which was used in the risk model without interest force.

2. MAIN RESULTS

Following [1], [2] consider the sequence

$$S_0 = x, \quad S_{n+1} = \xi_n S_n + (1 - \eta_{n+1}), \quad n \geq 0 \tag{1}$$

which represents the capital of an insurance company at discrete time moments $0, 1, \ldots$. Here x is the initial capital and $\eta_1, \ \eta_2, \ldots$ are claims arriving at the moments $1, 2, \ldots;$ $\xi_0, \ \xi_1, \ldots$ are the inflation factors (accumulation factors) in the periods from the 0 to 1, 1 to 2, etc. Assume that the company gets a unit capital during a unit time interval and the sequences $\eta_1, \ \eta_2, \ldots; \ \xi_0, \ \xi_1, \ldots$ are independent and consist of independent and identically distributed random variables (i.i.d.r.v.'s). In each sequence the r.v.'s have their own common distribution function and $\mathbf{P}(\xi_0 > 0) = 1$. Denote

$$\Psi^{(n)}(x) = \mathbf{P}(\min_{0 \leq k \leq n} S_k < 0), \quad n \geq 0.$$

It is clear that the functions $\Psi^{(0)}(x), \ \Psi^{(1)}(x), \ldots$ are monotonically nonincreasing in $x \geq 0$ and for each x they form a monotonically nondecreasing sequence in $n \geq 0$. So, there is the limit

$$\lim_{n \to \infty} \Psi^{(n)}(x) = \mathbf{P}(\inf_{0 \leq k} S_k < 0) = \Psi(x)$$

which is monotonically nonincreasing in $x \geq 0$.

Theorem 1. *Assume that*

$$q = E\xi_0 < 1 \tag{2}$$

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and for some $a > 0$

$$a(q-1)+1<0, \tag{3}$$

$$\mathbf{P}(\eta_1 > a) > 0 \tag{4}$$

then

$$\Psi(x) = 1, \quad x \geq 0. \tag{5}$$

Proof. Define the random sequences $R_0 = x, \quad R_{n+1} = \xi_n R_n + 1;$

$$G_0 = x, \quad G_{n+1} = \xi_n \max(G_n, a) + 1, \quad n \geq 0. \tag{6}$$

Using a formula from (1), by induction on n it is easy to prove that almost surely

$$S_n \leq (R_n - \eta_{n+1}) \leq G_n - \eta_{n+1}, \quad n \geq 0,$$

and consequently

$$\mathbf{P}\left(\inf_{n\geq 0} S_n < 0\right) \geq \mathbf{P}\left(\inf_{n\geq 0}(R_n - \eta_{n+1}) < 0\right) \geq \mathbf{P}\left(\inf_{n\geq 0}(G_n - \eta_{n+1}) < 0\right). \tag{7}$$

Denote by $0 \leq n_0 < n_1 = n_0 + m_0 < n_2 = n_1 + m_1 < \ldots$ the successive time moments which satisfy the equalities

$$G_{n_k} \leq a, \quad k \geq 0.$$

Condition (3) and formulas (6) imply that for $y > a$

$$E[(G_{n+1} - G_n)/G_n = y] = y(q-1)+1 \leq a(q-1)+1 < 0.$$

Consequently, the known results on the method of test functions (see [3], chapter 3, § 2) lead to

$$En_0 < \infty, \quad Em_0 < \infty, \quad Em_1 < \infty, \ldots$$

and hence,

$$\mathbf{P}(n_0 < \infty) = \mathbf{P}(m_0 < \infty) = \mathbf{P}(m_1 < \infty) = \ldots = 1.$$

Then

$$\mathbf{P}(n_1 < \infty) = \mathbf{P}(n_2 < \infty) = \ldots = 1.$$

The random sequences $n_0, n_1, \ldots; \quad \eta_1, \eta_2, \ldots$ are independent and the sequence $\eta_1, \eta_2, \ldots$ consists of i.i.d.r.v's. So, it is easy to prove that the random sequences

$$\eta_{n_0}, \eta_{n_1}, \ldots; \quad \eta_1, \eta_2, \ldots$$

are identically distributed. Consequently, formulas (7) and condition (4) give

$$\mathbf{P}\left(\inf_{n\geq 0} S_n < 0\right) \geq \mathbf{P}\left(\inf_{k\geq 0}(G_{n_k} - \eta_{n_k+1}) < 0\right) =$$

$$= \mathbf{P}\left(\inf_{k\geq 0}(a - \eta_{n_k+1}) < 0\right) = \mathbf{P}\left(\inf_{k\geq 1}(a - \eta_k) < 0\right) = 1.$$

The theorem 1 is proved.

Remark 1. Theorem 1 generalizes a result from [4] on the behavior of the ruin probability in a risk model under constant inflation to the case of stochastic inflation.

Theorem 2. *Assume that*

$$b = E\xi_0^{-1} < 1, \tag{8}$$

$$a = E\eta_1 < \infty \tag{9}$$

then

$$\lim_{x\to\infty} \Psi(x) = 0. \tag{10}$$

Proof. Define the random sequence

$$\tilde{S}_0 = x, \;\; \tilde{S}_{n+1} = x + \sum_{i=0}^{n} \left(\prod_{j=0}^{i} \xi_j^{-1} \right) (1 - \eta_{i+1}), \;\; n \geq 0.$$

It is obvious that

$$\Psi^{(n)}(x) = \mathbf{P}\left(\min_{0\leq k\leq n} \tilde{S}_k < 0 \right), \;\; \Psi(x) = \mathbf{P}\left(\inf_{0\leq k} \tilde{S}_k < 0 \right);$$

and

$$\Psi(x) = \mathbf{P}\left(\inf_{0\leq k} \left[x + \sum_{i=0}^{k-1} \left(\prod_{j=0}^{i} \xi_j^{-1} \right) (1 - \eta_{i+1}) \right] < 0 \right) =$$

$$= \mathbf{P}\left(\sup_{0\leq k} \sum_{i=0}^{k-1} \left(\prod_{j=0}^{i} \xi_j^{-1} \right) (\eta_{i+1} - 1) > x \right) \leq \mathbf{P}\left(\sup_{0\leq k} \sum_{i=0}^{k-1} \left(\prod_{j=0}^{i} \xi_j^{-1} \right) \eta_{i+1} > x \right) =$$

$$= \mathbf{P}\left(A > x \right),$$

where

$$A = \sum_{i=0}^{\infty} \left(\prod_{j=0}^{i} \xi_j^{-1} \right) \eta_{i+1}.$$

The conditions of the theorem 2 give:

$$EA \leq \frac{ba}{1-b} = c < \infty$$

and for each $x > 0$

$$\mathbf{P}(A > x) \leq \frac{c}{x} \to 0, \;\; x \to \infty.$$

So the theorem 2 is proved.

Remark 2. Theorem 2 generalizes some known results for classical risk model under constant interest force [5], [6], [7], [8] to stochastic interest force case.

Assume that $\xi_i = \xi_i(\lambda) = \exp(\lambda y_i), \;\; i \geq 0, \;\; -a \leq \lambda \leq a$, and the convex function $Y(\lambda) = E\xi_i(\lambda))$ is monotonically nondecreasing on the set $[-a, a]$. Then theorems 1, 2 and the ergodicity criterion for Lindley chains [9] (which describes a sequence of waiting times in one-server queueing system $G|G|1|\infty$) give

Theorem 3. *1. Assume that $\lambda < 0$ and for some $a = a(\lambda) > 0$ the conditions*

$$a(Y(\lambda) - 1) + 1 < 0$$

and (4) hold. Then (5) takes place.

2. Assume that $\lambda > 0$ and condition (9) hold. Then (10) takes place.

3. Assume that $\lambda = 0$. If $E\eta_1 \geq 1$, then (5) holds and if $E\eta_1 < 1$, then (10) takes place.

Denote: $\eta_{i+1} - 1 = \delta_i, \ \ i \geq 0$, by definition $\Psi^{(n)}(x) = \mathbf{P}\left(U_n > x\right)$ where

$$U_n = \max_{0 \leq k \leq n} \sum_{i=0}^{k-1} \left(\prod_{j=0}^{i} \xi_j^{-1} \right) \delta_i, \ \ n \geq 0. \tag{11}$$

Define the independent sequences $\{\tilde{\delta}_i, \ \ i \geq 0\}, \ \ \{\tilde{\xi}_i, \ \ i \geq 0, \}$ of i.i.d.r.v. by equalities

$$\tilde{\delta}_i \stackrel{d}{=} \eta_i - 1, \ \ \tilde{\xi}_i \stackrel{d}{=} \xi_i^{-1}. \tag{12}$$

Here $\stackrel{d}{=}$ means the equality of r.v.'s in distribution. Define the Markov chain

$$\tilde{U}_0 = 0, \ \ \tilde{U}_{n+1} = \tilde{\xi}_n \max(0, \ \tilde{\delta}_n + \tilde{U}_n), \ \ n \geq 0. \tag{13}$$

Theorem 4. *Then:*

$$\tilde{U}_m \stackrel{d}{=} U_m \tag{14}$$

and so

$$\Psi^{(m)}(x) = \mathbf{P}(\tilde{U}_m > x), \ \ m \geq 0.$$

Proof. For $m = 0$ the equality (14) is obvious. Assume that it is true for $m = n$ and prove it for $m = n + 1$. From (11) we obtain

$$U_n = \max(0, \ \xi_0^{-1}\delta_0, \ \xi_0^{-1}\delta_0 + \xi_0^{-1}\xi_1^{-1}\delta_1, \ldots, \xi_0^{-1}\delta_0 +$$

$$+ \xi_0^{-1}\xi_1^{-1}\delta_1 + \ldots + \xi_0^{-1} \ldots \xi_{n-1}^{-1}\delta_{n-1}). \tag{15}$$

In frames of equalities (12) choose $\tilde{\xi}_i = \xi_{n-i}^{-1}, \ \ \tilde{\delta}_i = \delta_{n-i}, \ 0 \leq i \leq n$, then from (13), (15) $U_{n+1} \stackrel{d}{=}$

$$\stackrel{d}{=} \max(0, \ \tilde{\xi}_n\tilde{\delta}_n, \ \tilde{\xi}_n\tilde{\delta}_n + \tilde{\xi}_n\tilde{\xi}_{n-1}\tilde{\delta}_{n-1}, \ldots, \tilde{\xi}_n\tilde{\delta}_n + \tilde{\xi}_n\tilde{\xi}_{n-1}\tilde{\delta}_{n-1} + \ldots + \tilde{\xi}_n \ldots \tilde{\xi}_0\tilde{\delta}_0) =$$
$$= \max(0, \ \tilde{\xi}_n \max(\tilde{\delta}_n, \ \tilde{\delta}_n + \tilde{\xi}_{n-1}\tilde{\delta}_{n-1}, \ldots, \tilde{\delta}_n + \ldots + \tilde{\xi}_{n-1} \ldots \tilde{\xi}_0\tilde{\delta}_0)) =$$
$$= \max(0, \ \tilde{\xi}_n(\tilde{\delta}_n + \max(0, \ \tilde{\xi}_{n-1}\tilde{\delta}_{n-1}, \ldots, \tilde{\xi}_{n-1}\tilde{\delta}_{n-1} + \ldots + \tilde{\xi}_{n-1} \ldots \tilde{\xi}_0\tilde{\delta}_0))) =$$

$= \tilde{U}_{n+1}$

The theorem 4 is proved.

Remark 3. Theorem 4 generalizes the Lindley chain for one-server queueing system $G|G|1|\infty$ to the risk model under stochastic interest force (1). This theorem gives convenient statistical simulation algorithms for the calculation of the ruin probability. Similarly to the well-known Embrechts-Veraverbeke formula [10], it allows to construct asymptotic formulas for the ruin probability on finite time interval.

3. CONCLUSION

The specific feature of the obtained results is the fact (see the theorem 3) that the limit behavior of ruin probability depends not on the claim distribution but on the distribution of the inflation factor.

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