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A NOTE ON LIMIT THEOREM FOR OVERLOADED PROCESSOR SHARING QUEUE

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Abstract—We give a new simple proof of the limit theorem for the overloaded M/G/1 processor sharing queue with egalitarian processor sharing (EPS). This theorem was announced in [1] as theorem 6.4.

1. INTRODUCTION

In 1990, Yashkov announced the theorem (see theorem 6.4 in [1]) for the limiting behaviour of the number of jobs $L(t)$ in the overloaded M/G/1 queue with the discipline of egalitarian processor sharing (EPS). The EPS discipline is frequently used approximation (and limit of) the round–robin scheduling algorithm used in computers. With the EPS discipline each job receives service at rate $1/n$ when there are n jobs in service. Now it is a basic model in queueing theory. Since 1990, a number of works have appeared with various proofs of this limit theorem and even its generalizations to the case GI/G/1—EPS (see, for example, the recent paper of Gromoll, Puha and Williams [2] and the references herein, and also [3]). All these proofs are rather difficult and cumbersome (some of them are based on concepts of fluid limit or on insufficiently explored theorems from branching processes theory). So, we believe that we have something to contribute.

The purpose of this note is to present a new simple proof of this limit theorem which, in fact, is a some version of the strong law of large numbers (SLLN). Our proof is based, in particular, on the (relatively new) results concerning the time–dependent queue–length distribution in the M/G/1—EPS queue [4, Ch.2].

The remainder of this note is organized as follows. Section 2 contains necessary preliminaries. The proof of the main result is given in section 3.

2. PRELIMINARIES

We consider an M/G/1 queue where the jobs arrive according to Poisson process with the rate λ. Let $B(x)$ be the distribution function of the service times (better: the lengths) of jobs $(B(0) = 0)$ with the mean $\beta_1 < \infty$ and the Laplace-Stieltjes transform (LST) $\beta(s)$. The service discipline is the egalitarian processor–sharing (EPS): when there are n jobs in the system, each receive service at a rate $\frac{1}{n}$ irrespective of its attained service. (An idea of EPS is due to Kleinrock [5]). Let $L(t)$ be the number of jobs at time t. We assume that the system is overloaded, in other words,

$$\rho = \lambda\beta_1 > 1.$$

We assume also that the system is empty at time $t = 0$. The large difficulties in the study of the EPS queue are caused by the same phenomenon as occuring in the analysis of the stationary sojourn time (see [6]): EPS allows jobs to overtake each other.

Let $\pi(s)$ be the LST of the busy period distribution $\Pi(x) = \mathsf{P}(\Pi \le x)$, i.e., it is the root of the functional equation

$$\pi(s) = \beta(s + \lambda - \lambda\pi(s)) \tag{2.1}$$

with the smallest absolute value. It is well–known that when $\rho > 1$, (2.1) has two solutions as $s = 0$: $\pi_1(0) = 1$ and $\pi_2(0) = \pi(0) < 1$. As $s > 0$, (2.1) has single solution $\pi(s)$, thereby $\pi(s) \to \pi(0)$ as $s \to 0$, hence $\Pi = \infty$ with probability $1 - \pi(0)$.

The study of the transient behaviour of the process of the number of jobs $\{L(t)\}$ leads to the following version of the SLLN [1, p.139] which holds for zero initial condition (the queue is empty at $t = 0$).

Theorem 1. *When $\rho > 1$ in the M/G/1—EPS queue, we have*

$$\lim_{t\to\infty} \frac{L(t)}{t} = \gamma \quad w.p.\,1 \tag{2.2}$$

where $\gamma = \lambda(1 - \pi(0)) > 0$ is the solution of the equation

$$\lambda \int_0^\infty \mathrm{e}^{-\gamma x}(1 - B(x))dx = 1. \tag{2.3}$$

The simple proof of this theorem is given in Section 3.

3. PROOF AND SOME COMMENTS

In the proof[1] we used the following three lemata.

Lemma 1. *If the process $\{L(t) : t \geq 0\}$ has stationary independent increments and there exists $\mathsf{E}[L(t)] = \gamma t$, then it holds the following SSLN:*

$$\lim_{t\to\infty} \frac{L(t)}{t} = \gamma \quad w.p.\,1\,. \tag{3.1}$$

Proof. It is the well–known result, see, e.g. [8, section 16.3.3]. □

Remark 1. The examples of such processes are the following: Poisson process, Brownian motion (Wiener process), compound Poisson process with the negative drift, Levy process, etc.

Lemma 2. *The process $\{L(t) : t \geq 0\}$ in the M/G/1–EPS queue has stationary independent increments as $\rho > 1$.*

Proof. The above follows as some by–product of the analysis in [6, section 3]. For additional details see also [1, p.116], [9, p.694]. □

Lemma 3. *$\mathsf{E}[L(t)]$ in the M/G/1–EPS queue for any ρ is given by its Laplace transform:*

$$\int_0^\infty \mathsf{E}[L(t)]\mathrm{e}^{-st}dt = \frac{\lambda(1 - \pi(s))}{s^2}\,, \tag{3.2}$$

where $\pi(s)$ is given by (2.1).

Proof. We take $\lim_{z\to 1} \frac{\partial g_0(s,z)}{\partial z}$, where $g_0(s, z)$ is the Laplace transform of the generating function for the $L(t)$ (it is easily). The double transform $g_0(s, z)$ was found in [4, formula (2.108)] (see also [1, theorem (2.18)]). □

[1] An abridged version of the proof is also contained in [7].

Proof. To prove the theorem it is sufficient now to combine the above propositions and invert the Laplace transform 3.2. Both these operations are done easily. □

Remark 2. The constant γ is called a Malthus parameter in the branching processes theory which as a rule does not use the concept of busy period. As opposed to this, the modern queueing theory is based, as a rule, on the concept of busy period, but it does not use the concept of the Malthus parameter. This is one of the examples of the difficulties which take place under the attempts to use some useful results from, say, the queueing theory in branching processes (and vice versa).

Remark 3. It is well known that for any work conserving discipline, the workload of the single server queue grows at rate $\rho - 1$ as $\rho > 1$ (the transient behaviour), see [10, p.633]. The same holds for the number of jobs in the overloaded M/G/1–FCFS queue. In other words, the equivalent of Theorem 1 for the FCFS discipline (as $\rho > 1$) is

$$\lim_{t\to\infty} \frac{L(t)}{t} = \rho - 1 \quad w.p.\,1.$$

Theorem 1 shows that the transient behaviour of the number of jobs in the M/G/1–EPS queue is not at all as simple as one could expect at first sight (as opposed to the familiar M/G/1—FCFS system).

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