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Priority Service of Primary Customers in the M/G/1/r Retrial Queueing System with Server Searching for Customers

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Abstract—In this paper a queueing system with a single customer searching server, retrials, finite buffer, Poisson flow and general distribution of service time is considered. An arriving customer joins a retrial orbit if he finds the buffer fully occupied. After finishing the service, the server searches for the next customer to be served among the customers in the buffer. The inter-retrial times are exponentially distributed with parameter γ. An orbiting customer can be served only if the buffer is free. A computational algorithm for stationary distribution of the primary queue length and a formula for the mean number of customers in the system are presented.

1. INTRODUCTION

In the last years interest in queueing systems with the server searching for customers is increasing. In fact these systems are able to model computer systems in which the processor has to spend a random time to find the next task to be processed. Despite the increasing interest, few results have been obtained for such systems.

The notion of search (at a service completion epoch) for customers has been introduced by Neuts and Ramalhoto in [1] in the context of classical $M/G/1$ queue. Recurrent formulas for stationary state probabilities and some performance indices have been obtained for a single server queueing system with two independent Poisson flows of customers and a general distribution function for service time in [2], where the searching for customers of different types is realized with non-preemptive priority. A retrial queueing system with a customer-searching server, a Poisson flow of customers and finite buffer has been analyzed in [3], where a computational algorithm for stationary state probabilities is derived. The results of [3] are generalized to the case of K Poisson flows of customers in [4]. The search for retrial customers for a $M/G/1$ queue with retrials has been studied in [5]. A $MAP/M/c$ queue with retrials and server searching for customers has been analyzed in [6]. The steady state analysis of the model has been performed using direct truncation and matrix-geometric approximation and efficient algorithm for calculation of performances indices has been presented. Matrix-geometric form for stationary state probabilities has been derived in [7] for the $PH/PH/1/r$ queueing system with search.

In this paper we consider a $M/G/1/r$ queueing system with retrials, server searching for customers in the buffer and priority service of primary customers. A similar queueing system without search has been investigated in [8], where a computational algorithm for stationary distribution of primary queue length and binomial moments of retrial queue length has been obtained. The present paper generalizes the results of [8] for the case in which the server realizes a searching for primary customers.

2. SYSTEM DESCRIPTION

We deal with a single server queueing system with a buffer of capacity r ($1 \leq r < \infty$) for primary customers and retrials.

Customers arrive from outside according to a Poisson flow with rate λ. The customer service times are independent and identically distributed random variables and they have a common arbitrary distribution function $B(x)$. We assume that $B(x)$ is absolutely continuous, that $B(0) = 0$ and

$$b = \int_0^\infty x dB(x) < \infty.$$

An arriving customer is allocated at the end of the queue in the buffer if there are free places, otherwise he joins the group of retrial customers which is called orbit. We suppose that the size of the orbit is infinite.

A customer waits in the buffer until the server searches for him. The duration of a search for customer is exponentially distributed with parameter θ. The time intervals between retrial requests of each orbiting customer are exponentially distributed with parameter γ.

If at the moment of repeated request the server is serving a customer, the orbiting customer will rejoin the orbit, otherwise if the buffer is free and so the server is in the searching phase, the orbiting customer will interrupt the search and will be served.

Using Kendall's notation, we shall denote this queueing system as M/G/1/r queue with retrials and search.

3. SYSTEM OF EQUILIBRIUM EQUATIONS

The behaviour in time of the system under consideration can be represented by the Markov process $\{\xi(t), t \geqslant 0\}$ with the state space

$$X = \left\{(i,n),\ (i,n,x),\quad n \geqslant 0,\ x \geqslant 0,\ i = \overline{0,r}\right\},$$

where the state (i,n) of the process $\xi(t)$ at the moment t means that the server is free, there are i customers in the primary queue and n customers in the orbit; $\xi(t) = (i,n,x)$ corresponds to the situation in which the queue includes i customers, the orbit contains n customers and the elapsed service time is equal to x.

Define

$$p^*_{in}(t) = \mathcal{P}\{\xi(t) = (i,n)\},$$
$$P_{in}(x,t) = \mathcal{P}\{\xi(t) \in \{(i,n) \times [0,x)\}.$$

We suppose that there exist stationary probabilities

$$p^*_{in} = \lim_{t\to\infty} p^*_{in}(t),$$
$$P_{in}(x) = \lim_{t\to\infty} P_{in}(x,t)$$

and stationary probability density $p_{in}(x) = P'_{in}(x)$. Let $p_{in} = P_{in}(\infty)$. Note that p_{in} is the stationary probability that i customers are in the primary queue and n customers are in the orbit. Let us denote

$$q_{in}(x) = \frac{p_{in}(x)}{1 - B(x)},\quad i = \overline{0,r},\ n \geq 0,$$
$$q_{in}(0) = q_{in},\quad i = \overline{0,r},\ n \geq 0.$$

It can be shown that the following system of equations holds:

$$\frac{d}{dx} q_{in}(x) = -\lambda q_{in}(x) + u(i)\lambda q_{i-1,n}(x),\quad i = \overline{0,r-1},\quad n \geq 0, \tag{3.1}$$

$$\frac{d}{dx} q_{rn}(x) = -\lambda q_{rn}(x) + \lambda q_{r-1,n}(x) + u(n)\lambda q_{r,n-1}(x),\quad n \geq 0, \tag{3.2}$$

$$0 = -\left(\lambda + n\gamma\right) p_{0n}^{*} + \int\limits_{0}^{\infty} q_{0n}\left(x\right) dB\left(x\right), \quad n \geq 0, \tag{3.3}$$

$$0 = -\left(\lambda + \theta\right) p_{in}^{*} + \lambda p_{i-1,n}^{*} + \int\limits_{0}^{\infty} q_{in}\left(x\right) dB\left(x\right), \quad i = \overline{1, r-1},\ n \geq 0, \tag{3.4}$$

$$0 = -\left(\lambda + \theta\right) p_{rn}^{*} + \lambda p_{r-1,n}^{*} + u\left(n\right) \lambda p_{r,n-1}^{*} + \int\limits_{0}^{\infty} q_{rn}\left(x\right) dB\left(x\right), \quad n \geq 0, \tag{3.5}$$

$$q_{0n}\left(0\right) = \gamma\left(n+1\right) p_{0,n+1}^{*} + \theta p_{1n}^{*}, \quad n \geq 0, \tag{3.6}$$

$$q_{in}\left(0\right) = \theta p_{i+1,n}^{*}, \quad i = \overline{1, r-1}, \quad n \geq 0, \tag{3.7}$$

$$q_{rn}\left(0\right) = 0, \quad n \geq 0, \tag{3.8}$$

where $u\left(x\right)$ is a unit Heavyside function.

4. SYSTEM OF EQUATIONS FOR FACTORIAL MOMENTS AND ITS SOLUTION

In this section we shall get the system of equations for factorial moments and its solution. For this purpose let us derive some basic relations that we shall use later.

Summing the equation (3.1) over $i = \overline{0, r-1}$ and adding (3.2), we obtain

$$\frac{d}{dx} q_{\cdot,n}\left(x\right) = -\lambda q_{rn}\left(x\right) + u\left(n\right) \lambda q_{r,n-1}\left(x\right), \quad n \geq 0, \tag{4.1}$$

where the central dot “ $\cdot$ ” will denote summing over all possible values of a discrete argument.

Summing (3.4) over i and adding (3.3), (3.5), we obtain

$$-\theta p_{\cdot,n}^{*} + \theta p_{0,n}^{*} + \int\limits_{0}^{\infty} q_{\cdot,n}\left(x\right) dB\left(x\right) - n\gamma p_{0n}^{*} - \lambda p_{rn}^{*} + u\left(n\right) \lambda p_{r,n-1}^{*} = 0. \tag{4.2}$$

Summing (3.7) over i and adding (3.6), (3.8) we have

$$\theta p_{\cdot,n}^{*} - \theta p_{0,n}^{*} = q_{\cdot,n}\left(0\right) - \gamma\left(n+1\right) p_{0,n+1}^{*}. \tag{4.3}$$

Substituting (4.3) into (4.2) we deduce that

$$q_{\cdot,n}\left(0\right) = -n\gamma p_{0n}^{*} + \gamma\left(n+1\right) p_{0,n+1}^{*} + \int\limits_{0}^{\infty} q_{\cdot,n}\left(x\right) dB\left(x\right) - \lambda p_{rn}^{*} + u\left(n\right) \lambda p_{r,n-1}^{*}, \quad n \geq 0. \tag{4.4}$$

By multiplying (4.1) by $1 - B\left(x\right)$, integrating it over x in $[0, +\infty)$, and using (4.4), we obtain

$$\gamma\left(n+1\right) p_{0,n+1}^{*} = \lambda p_{rn} + \lambda p_{rn}^{*}, \quad n \geq 0. \tag{4.5}$$

Now let us define, for $|z| \leq 1$, generating functions of $q_{in}\left(x\right)$, p_{in}^{*}, p_{in}:

$$Q_{i}\left(x, z\right) = \sum_{n=0}^{\infty} q_{in}\left(x\right) z^{n}, \quad P_{i}^{*}\left(z\right) = \sum_{n=0}^{\infty} p_{in}^{*} z^{n}, \quad P_{i}\left(z\right) = \sum_{n=0}^{\infty} p_{in} z^{n}, \quad i = \overline{0, r}.$$

Using the above functions, system of equations (3.1)−(3.8) takes the following form:

$$\frac{\partial}{\partial x} Q_i(x,z) = -\lambda Q_i(x,z) + u(i)\,\lambda Q_{i-1}(x,z), \quad i = \overline{0, r-1}, \tag{4.6}$$

$$\frac{\partial}{\partial x} Q_r(x,z) = -\lambda (1-z)\, Q_r(x,z) + \lambda Q_{r-1}(x,z), \tag{4.7}$$

$$0 = -\lambda P_0^*(z) - \gamma z \frac{d}{dz} P_0^*(z) + \int_0^\infty Q_0(x,z)\, dB(x), \tag{4.8}$$

$$0 = -(\lambda + \theta)\, P_i^*(z) + \lambda P_{i-1}^*(z) + \int_0^\infty Q_i(x,z)\, dB(x), \quad i = \overline{1, r-1}, \tag{4.9}$$

$$0 = -\lambda \left(1 - z + \frac{\theta}{\lambda}\right) P_r^*(z) + \lambda P_{r-1}^*(z) + \int_0^\infty Q_r(x,z)\, dB(x), \tag{4.10}$$

$$Q_0(z) = \gamma \frac{d}{dz} P_0^*(z) + \theta P_1^*(z), \tag{4.11}$$

$$Q_i(z) = \theta P_{i+1}^*(z), \quad i = \overline{1, r-1}, \tag{4.12}$$

$$Q_r(z) = 0. \tag{4.13}$$

It is clear that solving differential equations (4.6), (4.7) with boundary conditions (4.8)−(4.13) is fairly difficult from an analytic point of view. It is simpler to search only for the factorial moments of the number of customers in the orbit. In order to do it, let us define

$$N_{im}(x) = \sum_{n=m}^{\infty} (n)_m\, q_{in}(x), \quad N_{im}^* = \sum_{n=m}^{\infty} (n)_m\, p_{in}^*, \quad N_{im} = \sum_{n=m}^{\infty} (n)_m\, p_{in}, \quad i = \overline{0, r}, \quad m \geq 0, \tag{4.14}$$

where $(n)_m = n(n-1)\ldots(n-m+1)$.

Differentiating m times the equations (4.6)−(4.13) over z and letting $z = 1$, we have

$$\frac{d}{dx} N_{im}(x) = -\lambda N_{im}(x) + u(i)\,\lambda N_{i-1,m}(x), \quad i = \overline{0, r-1}, \tag{4.15}$$

$$\frac{d}{dx} N_{rm}(x) = \lambda N_{r-1,m}(x) + \lambda m N_{r,m-1}(x), \tag{4.16}$$

$$0 = -(\lambda + \gamma m)\, N_{0m}^* - \gamma N_{0,m+1}^* + \int_0^\infty N_{0m}(x)\, dB(x), \tag{4.17}$$

$$0 = -(\lambda + \theta)\, N_{im}^* + \lambda N_{i-1,m}^* + \int_0^\infty N_{im}(x)\, dB(x), \quad i = \overline{1, r-1}, \tag{4.18}$$

$$0 = -\theta N_{rm}^* + \lambda N_{r-1,m}^* + \lambda m N_{r,m-1}^* + \int_0^\infty N_{rm}(x)\, dB(x), \tag{4.19}$$

$$N_{0m}(0) = \gamma N_{0,m+1}^* + \theta N_{1m}^*, \tag{4.20}$$

$$N_{im}(0) = \theta N^*_{i+1,m}, \quad i = \overline{1, r-1}, \tag{4.21}$$

$$N_{rm}(0) = 0. \tag{4.22}$$

Note that we can rewrite (4.5), using (4.20)–(4.22), as

$$\gamma N^*_{0,m+1} = \lambda\left(N_{rm} + N^*_{rm}\right). \tag{4.23}$$

This relation will be useful later.

Now we have to analyze the system (4.15)–(4.22).

By multiplying equation (4.15) by $1 - B(x)$ and integrating it over x in $[0, +\infty)$, we obtain

$$-N_{im}(0) + \int\limits_0^\infty N_{im}(x)\, dB(x) = -\lambda N_{im} + u(i)\,\lambda N_{i-1,m}, \quad i = \overline{0, r-1}. \tag{4.24}$$

Substituting (4.24) for $i = 0$ into (4.17) and (4.24) into (4.18), we obtain respectively

$$\begin{aligned} N_{0m}(0) &= (\lambda + \gamma m)\, N^*_{0m} + \gamma N^*_{0,m+1} + \lambda N_{0m}, \\ N_{im}(0) &= (\lambda + \theta)\, N^*_{im} - \lambda N^*_{i-1,m} + \lambda N_{im} - u(i)\,\lambda N_{i-1,m}, \quad i = \overline{1, r-1}. \end{aligned} \tag{4.25}$$

From (4.25), using (4.20), we have

$$N_{im}(0) = \lambda\left(N_{im} + N^*_{im}\right) + \gamma m N^*_{0m}, \quad i = \overline{1, r-1}. \tag{4.26}$$

Summing (4.26) over $i = \overline{1, r-1}$, using (4.25), (4.22) and (4.23), we have

$$N_{\cdot,m}(0) = \lambda\left(N_{\cdot,m} + N^*_{\cdot,m}\right) + r\gamma m N^*_{0m}, \tag{4.27}$$

from which

$$N_m = \frac{1}{\lambda} N_m(0) - \frac{rm}{\widehat{\rho}} N^*_{0m}, \quad m \geq 0, \tag{4.28}$$

where $N_m(0) = N_{\cdot,m}(0)$, $N_m = N_{\cdot,m} + N^*_{\cdot,m}$, and $\widehat{\rho} = \frac{\lambda}{\gamma}$.

Now we shall find $N_m(0)$ and N^*_{0m}, in order to obtain N_m.

First of all we can easily show that for fixed m the solution of equation (4.15) is

$$N_{im}(x) = e^{-\lambda x} \sum_{k=0}^{i} \frac{(\lambda x)^k}{k!} N_{i-k,m}(0), \quad i = \overline{0, r-1}.$$

Substituting this solution into equations for boundary conditions (4.17), (4.18), we obtain respectively

$$\begin{aligned} 0 &= -(\lambda + \gamma m)\, N^*_{0m} - \gamma N^*_{0,m+1} + N_{0m}(0)\,\beta_0, \\ 0 &= -(\lambda + \theta)\, N^*_{im} + \lambda N^*_{i-1,m} + \sum_{k=0}^{i} \beta_k N_{i-k,m}(0), \quad i = \overline{1, r-1}, \end{aligned}$$

where $\beta_k = \int\limits_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} dB(x)$, $\quad k = \overline{0, r-1}$.

From these relations, using (4.20), (4.21), and (4.23), introducing the notations

$$A_m = \left(N_{rm} + N^*_{rm}\right), \tag{4.29}$$

we have

$$\begin{aligned} &\beta_0\theta N^*_{1m} = (\lambda+\gamma m)\, N^*_{0m} + \lambda\,(1-\beta_0)\, A_m, \\ &\theta\sum_{k=0}^{i}\beta_k N^*_{i-k+1,m} - (\lambda+\theta)\, N^*_{im} + \lambda N^*_{i-1,m} = -\lambda\beta_i A_m, \quad i=\overline{1,r}. \end{aligned} \tag{4.30}$$

The matrix of coefficients for the system (4.30) has a triangular structure; so its solution can be represented as

$$N^*_{im} = \varphi_i N^*_{0m} + \sigma_i A_m, \quad i=\overline{1,r}, \quad m\geq 0, \tag{4.31}$$

where φ_i and σ_i are calculated as follows

$$\varphi_0=1, \quad \varphi_1=\tfrac{1}{\theta\beta_0}\,(\lambda+\gamma m)\,, \quad \varphi_i=\tfrac{1}{\theta\beta_0}\left[(\lambda+\theta)\,\varphi_{i-1}-\lambda\varphi_{i-2}-\theta\sum_{k=1}^{i-1}\varphi_k\beta_{i-k}\right], \quad i=\overline{2,r},$$
$$\sigma_0=0, \quad \sigma_1=\tfrac{\lambda(1-\beta_0)}{\theta\beta_0}, \quad \sigma_i=\tfrac{1}{\theta\beta_0}\left[(\lambda+\theta)\,\sigma_{i-1}-\lambda\sigma_{i-2}-\lambda\beta_{i-1}-\theta\sum_{k=1}^{i-1}\sigma_k\beta_{i-k}\right], \quad i=\overline{2,r}.$$

From the relation (4.31), we have

$$N^*_{im} = \chi_i N^*_{0m} + \psi_i N_{rm}, \quad i=\overline{1,r}, \tag{4.32}$$

where

$$\chi_i = \varphi_i + \frac{\sigma_i\varphi_r}{1-\sigma_r}, \quad i=\overline{1,r},$$
$$\psi_i = \frac{\sigma_i}{1-\sigma_r}, \quad i=\overline{1,r}.$$

From (4.20), (4.21), and using (4.32) for $i=1$, $i=r$ and (4.23), we obtain

$$N_{im}\,(0) = f_i N^*_{0m} + g_i N_{rm}, \qquad i=\overline{0,r-1}, \tag{4.33}$$

where

$$f_0=\lambda\chi_r+\theta\chi_1, \quad f_i=\theta\chi_{i+1}, \qquad i=\overline{1,r-1},$$

and

$$g_0=\lambda\psi_r+\lambda+\theta\psi_1, \quad g_i=\theta\psi_{i+1}, \qquad i=\overline{1,r-1}.$$

In order to obtain $N_{im}\,(0)$ we have to calculate N^*_{0m} and N_{rm}. First we shall obtain the latter in the case $m=0$. From (4.14) in which we put $m=0$, summing over i

$$N_{\cdot,0}\,(x) = q_{\cdot,\cdot}\,(x) = q_\cdot(x), \quad N^*_{\cdot,0} = p^*_{\cdot,\cdot}, \quad N_{\cdot,0} = p_{\cdot,\cdot}\,, \tag{4.34}$$

hence, using the normalizing condition, from (4.27) for $m=0$ we have

$$\lambda = q_\cdot, \tag{4.35}$$

where $q_\cdot = q_\cdot\,(0)$.

From (4.15) and (4.16) for $m=0$, using (4.34), we have

$$q_\cdot\,(x) = q_\cdot\ . \tag{4.36}$$

By multiplying (4.36) by $1-B\,(x)$ and integrating it over x in $[0,+\infty)$, we obtain

$$p_{\cdot,\cdot} = bq_\cdot\ . \tag{4.37}$$

From (4.35) and (4.37) we get

$$p_{\cdot,\cdot} = \rho, \tag{4.38}$$

where $\rho = \lambda b$. From (4.38), and from normalizing condition we have

$$p^*_{\cdot,\cdot} = 1 - \rho. \tag{4.39}$$

Note that the relation (4.39) implies that the condition $\rho < 1$ is a necessary ergodicity condition for the Markov process under consideration. One can show that this condition is a sufficient one.

Further, summing (4.33) over $i = \overline{1, r-1}$ for $m = 0$ and using (4.35), we obtain

$$\widetilde{f} p^*_{0,\cdot} + \widetilde{g} p_{r,\cdot} = \lambda, \tag{4.40}$$

where $\widetilde{f} = \sum\limits_{i=0}^{r-1} f_i, \quad \widetilde{g} = \sum\limits_{i=0}^{r-1} g_i$.

Summing (4.32) for $m = 0$ over $i = \overline{1, r}$ and using (4.39) and (4.40), we get

$$p^*_{0,\cdot} = \frac{\lambda\psi - \widetilde{g}\,(1-\rho)}{\widetilde{f}\psi - (1+\chi)\,\widetilde{g}}, \quad p_{r,\cdot} = \frac{\widetilde{f}\,(1-\rho) - \lambda\,(\chi+1)}{\widetilde{f}\psi - (1+\chi)\,\widetilde{g}}.$$

From (4.32) for $m = 0$, and from (4.33) we obtain

$$p^*_{i,\cdot} = \chi_i p^*_{0,\cdot} + \psi_i p_{r,\cdot}, \quad i = \overline{1, r},$$
$$q_i = f_i p^*_{0,\cdot} + g_i p_{r,\cdot}, \quad i = \overline{0, r-1},$$

where $q_i = q_{i,\cdot}$.

From (4.25), (4.26), for $m = 0$ we find

$$p_{0,\cdot} = \frac{1}{\lambda} q_{0,\cdot} - p^*_{0,\cdot} - \frac{1}{\widehat{\rho}} N^*_{01}, \quad p_{i,\cdot} = \frac{1}{\lambda} q_{i,\cdot} - p^*_{i,\cdot}, \qquad i = \overline{1, r-1},$$

where N^*_{01} is determined by the formula

$$N^*_{01} = \widehat{\rho}\pi$$

and

$$\pi = p_{r,\cdot} + p^*_{r,\cdot}$$

which follows from (4.23) for $m = 0$.

Note that π is the probability that a new arrived customer will join the retrial queue.

To obtain the factorial moments for $m \geq 1$, we have to solve the equation (4.7). The solution of (4.7) is given in the following form:

$$Q_r\,(x,z) = \sum_{k=0}^{r-1} \frac{1}{z^{k+1}} \left[Q_{r-k-1}\,(z)\, e^{-(\lambda-\lambda z)x} - Q_{r-k-1}\,(x,z)\right].$$

We can rewrite this equality as follows:

$$\sum_{i=0}^{r} z^i Q_i\,(x,z) = \sum_{i=0}^{r-1} z^i Q_i\,(z)\, e^{-(\lambda-\lambda z)x}. \tag{4.41}$$

By multiplying (4.41) by $1 - B(x)$ and integrating it over x in $[0, +\infty)$, we have

$$\sum_{i=0}^{r} z^i P_i\,(x,z) = \frac{1-\beta\,(\lambda-\lambda z)}{\lambda\,(1-z)} \sum_{i=0}^{r-1} z^i Q_i\,(z)\,. \tag{4.42}$$

Let us denote

$$F(z)=\frac{1-\beta\left(\lambda-\lambda z\right)}{\lambda\left(1-z\right)}. \tag{4.43}$$

Differentiating m times the equation (4.42), taking into account (4.43) and putting $z=1$, we have

$$\sum_{i=0}^{r}\sum_{j=0}^{m}\binom{m}{j}\left(i\right)_{m-j}N_{ij}=\sum_{i=0}^{r-1}\sum_{j=0}^{m}\binom{m}{j}F^{(m-j)}(1)\sum_{k=0}^{j}\binom{j}{k}\left(i\right)_{j-k}N_{ik}\left(0\right), \quad m=\overline{0,r-1}. \tag{4.44}$$

Now we can calculate factorial moments N_{im} and N^*_{im}. We illustrate the computing procedure for the case $m=1$.

From (4.44) for $m=1$, taking into account (4.38), we have

$$N_1+\widetilde{N}_1=\rho+\frac{\lambda^2b^{(2)}}{2}+b\left[N_1\left(0\right)+Q\left(0\right)\right], \tag{4.45}$$

where

$$b^{(2)}=\int\limits_0^\infty x^2dB(x), \quad \widetilde{N}_1=\sum_{i=0}^{r}\left(i+1\right)p_{i,\cdot}, \quad Q\left(0\right)=\sum_{i=1}^{r-1}iq_i.$$

In turn, from (4.45) and (4.28) for $m=1$, we obtain the expression for $N_1\left(0\right)$:

$$N_1\left(0\right)=\frac{\lambda}{1-\rho}\left[\rho+\frac{\lambda^2b^{(2)}}{2}+bQ\left(0\right)+\frac{r}{\widehat{\rho}}N^*_{01}-\widetilde{N}_1\right]. \tag{4.46}$$

Thus, from (4.46) and (4.28) for $m=1$, we can calculate the values of $N_1\left(0\right)$ and N_1.

Summing (4.33) over $i=\overline{0,r-1}$, we have

$$N_1\left(0\right)=\widetilde{f}N^*_{01}+\widetilde{g}N_{r1}.$$

From here we can calculate N_{r1}. After this from (4.33) we obtain $N_{i1}\left(0\right), i=\overline{0,r-1}$ (these values are not necessary for this case, but they will be used in the case $m=2$).

Further, from (4.32) we get N^*_{i1}, $i=\overline{1,r}$. Then we can compute $N_{\cdot,1}$:

$$N_{\cdot,1}=N_1-N^*_{\cdot,1}.$$

In such a way we can calculate the mean number of primary customers in the buffer, N_1^{buf}, the mean number of customers waiting in the buffer and being served in the server, N_1^{sys}, and the mean number of retrial customers, N_1^{ret}, by the following formulas:

$$\begin{array}{ll} N_1^{buf}=\widetilde{N}_1+\widetilde{N_1^*}-\rho, & \widetilde{N_1^*}=\sum\limits_{i=1}^{r}ip^*_{i,\cdot}, \\ N_1^{sys}=N_1^{buf}+\rho, & N_1^{ret}=N_{\cdot,1}+N^*_{\cdot,1}. \end{array} \tag{4.47}$$

The mean number of customers in the system, N, is defined as follows:

$$N=N_1^{sys}+N_1^{ret}.$$

5. MEAN NUMBER OF CUSTOMERS IN THE SYSTEM

In this section we shall find another expression for the mean number N of customers in the system, based on some other arguments.

Let

$$p_i = p_{i\cdot} + p^*_{i,\cdot}, \quad i = \overline{0, r}, \tag{5.1}$$

be the probability that the primary queue contains i customers.

From (4.26) for $m = 0$, it follows that

$$p_i = \frac{q_i}{\lambda}, \quad i = \overline{1, r-1}. \tag{5.2}$$

From (4.47), using (5.1) and (4.38), we have

$$N = \rho + \sum_{i=0}^{r} i p_i + N_1^{ret}. \tag{5.3}$$

Define

$$Q(x) = \sum_{i=1}^{r} i q_{i,\cdot}(x).$$

From (4.15), (4.16) for $m = 0$,we obtain

$$\frac{d}{dx} q_{i,.}(x) = -\lambda q_{i,.}(x) + u(i)\,\lambda q_{i-1,.}(x), \quad i = \overline{0, r-1}, \tag{5.4}$$

$$\frac{d}{dx} q_{r,.}(x) = \lambda q_{r-1,.}(x). \tag{5.5}$$

Summing (5.4) over $i = \overline{0, r-1}$ multiplied by i and adding (5.5) multiplied by r, using (4.36) we get

$$\frac{d}{dx} Q(x) = \lambda q_{.} - \lambda q_{r,.}(x). \tag{5.6}$$

Summing (4.15) over $i = \overline{0, r-1}$ for $m = 1$ and adding (4.16) for $m = 1$, it follows that

$$\frac{d}{dx} N_{.,1}(x) = \lambda N_{r0}(x) = \lambda q_{r,.}(x). \tag{5.7}$$

By substituting (5.7) into (5.6) and using (4.35) we obtain

$$Q(x) + N_{.,1}(x) = \lambda^2 x + Q(0) + N_{.,1}(0). \tag{5.8}$$

By multiplying (5.8) by $1 - B(x)$, integrating it over x in $[0, +\infty)$ we have

$$N = \rho + \frac{\lambda^2}{2} b^{(2)} + b\left[Q(0) + N_{.,1}(0)\right] + \widetilde{N_1^*} + N^*_{.,1}. \tag{5.9}$$

By multiplying (5.2) by i, summing it over $i = \overline{0, r-1}$, and using (5.1), and (4.38), we get

$$\frac{1}{\lambda} Q(0) = \sum_{i=0}^{r} (i+1)\, p_{i,.} + \sum_{i=0}^{r} i p^*_{i,.} - \rho - r p_{r\cdot}. \tag{5.10}$$

Summing (5.10), (4.28) and using (5.3), and (4.23) for $m = 0$ and $m = 1$, we obtain

$$N = \rho + \frac{1}{\lambda}\left[Q(0) + N_1(0)\right].$$

From here, using (5.9) and (4.23), it follows that

$$N = \rho + \frac{b^2\left(1 + c_B^2\right)}{2(1-\rho)} + \frac{\widetilde{N_1^*} + N^*_{.,1}}{1-\rho},$$

where c_B is the coefficient of variation of $B(x)$.

6. NUMERICAL RESULTS

In order to illustrate the results obtained, we present some numerical examples computed by MATHEMATICA 4.0. Results are given in Examples 6.1 and 6.2, where p_i is the probability that primary queue contains i customers; N_1^{buf} is the mean number of primary customers in the buffer, and N is the mean number of customers in the system.

Example 1. We consider the case in which the distribution function $B(x)$ is exponentially distributed with parameter μ, and the following values of the system parameters:

$$r = 15, \lambda = 1, \mu = 2, \theta = 5.$$

Results are given in the Table 1:

Table 1

γ	p_0	p_r	N_1^{buf}	N
5	0.450269	0.000336411	1.48765	2.49551
10	0.450269	0.000336411	1.48765	2.4955
20	0.450269	0.000336411	1.48765	2.49549
50	0.450269	0.000336411	1.48765	2.49548

The table shows the independence of the system performance characteristics on the parameter γ This is due to priority service of primary customers.

Example 2. In this example, we illustrate the influence of parameter θ on the system performance. We consider again the exponential service time and the following values for the parameters of the system:

$$r = 15, \lambda = 1, \gamma = 0.3.$$

Numerical results are reported in Table 2 for $b = 0.5$ and in Table 3 for the case $b = 0.9$.

Table 2

$1/\Theta$	p_0	p_r	N_1^{buf}	N
$\frac{1}{10}b$	0.67501	0.000017	0.666321	1.58332
$\frac{1}{100}b$	0.74250	0	0.515017	1.50758
$\frac{1}{1000}b$	0.74925	0	0.501378	1.50075

As intuition may suggest, Table 2 and Table 3 show that when the customer searching time decreases, the performances of the system improve.

Table 3

$1/\theta$	p_0	p_r	N_1^{buf}	N
$\frac{1}{10}b$	0.07532	0.0525	7.17001	18.4582
$\frac{1}{100}b$	0.1956	0.0248	5.03042	17.1822
$\frac{1}{1000}b$	0.2089	0.0229	4.84498	17.1078

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