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MATHEMATICAL MODELS

SOME INSIGHT TO THE TIME–DEPENDENT PROPERTIES OF THE QUEUE LENGTH PROCESS IN THE M/G/1–EPS and LCFS–P QUEUES[1]

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Abstract—We give an additional (to [4], [6]) insight to the properties of the time–dependent queue–length distribution in the M/G/1 queue with the Egalitarian Processor Sharing (EPS) and Preemptive Last Come–First Served (LCFS–P) disciplines under the zero initial condition. We put forward also some new conjectures.

1. INTRODUCTION AND PRELIMINARIES

It is well known, due to Sakata et al. [1] (see also [2]), that the stationary distribution $(\mathrm{P_n})_{\mathrm{n}\geq 0}$ of the number of jobs in the M/G/1 queue with the egalitarian processor sharing (EPS) is geometrically distributed

$$\mathrm{P_n} = (1-\rho)\rho^{\mathrm{n}}, \quad \mathrm{n} = 0, 1, \ldots, \tag{1.1}$$

where $\rho = \lambda \int_0^\infty (1 - B(x))dx < 1$. Here λ is the Poisson input rate and $B(x)$ is the distribution function of the service time. Besides, $(\mathrm{P_n})_{\mathrm{n}\geq 0}$ depends on the service time only through its mean.

The determination of the the time–dependent distribution of the queue length in this queue was much more hard problem, see [3], [4], [6], [5]. Since 1988–1989 its non-trivial solution is known from these works. It was obtained by means of the further development of the new analytical method from [7] which allowed at first to solve the open (stationary) sojourn time problem for the M/G/1—EPS queue. (Later we found also the joint time–dependent distribution of the queue length at time t and of the sojourn time for the job arriving at time t [8], [9], but now we discuss only the time–dependent properties of the queue–length process.)

As we mentioned, we consider an M/G/1 queue where the jobs arrive according to Poisson process with the rate λ. Let $B(x)$ be the distribution function of the service times of jobs $(B(0) = 0, B(\infty) = 1)$ with the mean $\beta_1 < \infty$ and the Laplace–Stieltjes transform (LST) $\beta(s)$. The service discipline is the EPS: when there are n jobs in the system, each receive service at a rate $1/n$. Let $L(t)$ be the number of jobs at time t. We assume that the system is empty at time $t = 0$. Let $\zeta = \inf(t > 0 : L(t) = 0)$ and $\pi(s) = \mathsf{E}[\mathrm{e}^{-s\zeta}]$ is the LST of the busy period distribution, i.e., it is the positive root of the functional equation

$$\pi(s) = \beta(s + \lambda - \lambda\pi(s)) \tag{1.2}$$

with the smallest absolutely value. Our primary interest is the (non–Markovian) process $\{L(t),\ t \geq 0\}$ of the number jobs.

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Definition. The function

$$g_0(z,s) = \int_0^\infty \mathrm{e}^{-st}\mathsf{E}[z^{L(t)}|L(0)=0]\,dt, \quad \mathrm{Re}s > 0,\ |z| \le 1 \tag{1.3}$$

defines the Laplace transform w.r.t. t of the probability generating function for the time–dependent queue–length distribution $\mathrm{P}(L(t)=i|L(0)=0)$, $i=0,1,2,\ldots$.

It is known (see, for example, [4], [6]) that for the M/G/1–EPS queue

$$g_0(z,s) = \frac{1}{s+\lambda(1-z)(1-\pi(s))}, \tag{1.4}$$

where $\pi(s)$ is given by (1.2).

We intend to give an additional insight to the properties of this time–dependent distribution. There is a opinion that, outside the standard M/M/1 queue, the time–dependent solutions have been found only in special cases (see, for example, Jaiswal [10]) and ones involve most often doubly transform (as in (1.4)) which provide very little insight in the behaviour of the queueing probabilities $\mathrm{P_i}(t)$. We show that this such opinion may be false in our cases.

We shall use the well-known property of Laplace transforms: sampling at an independent exponentially distributed random time is equivalent to taking a Laplace transform w.r.t. time. This fact is actually used in the method of collective marks in queuing theory, introduced by van Dantzig [11].

Rest of this note is divided into two sections. In section 2, the main results are presented. In section 3, we provide short concluding remark.

2. MAIN RESULTS

Let T be the exponentially distributed random variable with the parameter $s>0$. Then the LHS of (1.4) (the double transform of $L(t)$) can be rewritten as $\mathsf{E}[z^{L(T)}]/s$ (here the numerator is the ordinary generating function that is sampling at exponentially distributed time T with the parameter s). Now it follows from (1.4) that

$$\mathsf{E}[z^{L(T)}] = \sum_{n=0}^{\infty} \frac{s}{s+\lambda-\lambda\pi(s)} \left(\frac{\lambda-\lambda\pi(s)}{s+\lambda-\lambda\pi(s)}\right)^n z^n. \tag{2.1}$$

We recall the definition

$$\mathsf{P}_{00}(t) \doteq \mathsf{P}(L(t)=0|L(0)=0). \tag{2.2}$$

Let $\tilde{p}_{00}(s)$ be the Laplace transform of $\mathsf{P}_{00}(t)$ w.r.t. t. It is well-known that it holds for all work–conserving M/G/1 queues under zero initial condition

$$\tilde{p}_{00}(s) = \frac{1}{s+\lambda-\lambda\pi(s)}. \tag{2.3}$$

We note also that

$$1 - s\tilde{p}_{00}(s) = \frac{\lambda-\lambda\pi(s)}{s+\lambda-\lambda\pi(s)}. \tag{2.4}$$

It is important that the RHS of (2.4) coincides with the RHS of the formula (2.14) in [12].

The formula (2.1) says that $L(T)$ is a geometrically distributed random variable with the probability of success $s\tilde{p}_{00}(s)$ (which coincides with the probability of termination of the corresponding terminating renewal process [12]). We note that, contrary to the stationary case, $\mathsf{E}[\mathsf{z}^{\mathsf{L(T)}}]$ depends on entire $B(x)$.

Remark 2.1. The queue-length process in the standard M/G/1—LCFS–P queue has the same time–dependent properties because the formula (1.4) holds also for this discipline (see Remark 3.5 in [13]). This last result (concerning the LCFS–P) is also reflected in rather hidden form in [4].

Recently, it was established some relation between the process $\{L(t),\ t \geq 0\}$ (only for the case of the discipline LCFS–P) and a Galton–Watson branching process (see Limic [14]) and even for more general case GI/G/1. Using this Limic's observation and our results from [5], [6], [8], [9], the main above results have been rediscovered by the Israel–Holland school in this year. Obviously, the time–dependent solutions in [4] outstripped the solutions mentioned at least on 16 years.

Remark 2.2. The EPS and LCFS–P disciplines belong to the class of positionally balanced disciplines [2], [4], [5] which, in turn, includes more narrow class of symmetric Kelly's disciplines. It is known that any discipline from the class of positionally balanced disciplines in the M/G/1 queue tranforms the Poisson input into Poisson output in the stationary mode and, besides, it holds the formula (1.1) above. The combination of the results from [2] (see also [4], [5]) allows us to put for forward the following hypothesis: the time–dependent distribution of the process $\{L(t),\ t \geq 0\}$ in the M/G/1 queue with any discipline from the class of positionally balanced disciplines ought to have the form which is given by the formula (1.4). Besides, it is reasonable to conjecture that the output process (in the time–dependent, that is, in the non–stationary case) ought to have some, possibly, simple and new properties which are not attributed to the general output process from the M/G/1 queue with any work–conserving (but not positionally balanced) discipline.

Remark 2.3. Due to Remark 2.1, the results of [5, Theorem 6.4], [15] concerning $\lim_{t\to\infty} L(t)/t$ as $\rho > 1$ for the M/G/1–EPS queue are also extended to the LCFS–P discipline, and, possibly, on any discipline from the class of positionally balanced disciplines (for the zero initial condition).

3. CONCLUSION

We pointed out some new time–dependent properties of the queue–length process in the M/G/1 queue with the processor sharing and LCFS–P disciplines. We put forward also new conjectures.

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