Информационные процессы, Том 5, № 4, 2005, стр. 283–300.

Rotation numbers of discontinuous orientation-preserving circle maps revisited

V. S. Kozyakin[1]

Institute for Information Transmission Problems
Russian Academy of Sciences
Received August 24, 2005

Abstract—The theory of circle homeomorphisms has a great number of deep results. However, sometimes continuity or single-valuedness of a circle map may be restrictive in theoretical constructions or applications. In this paper it is shown that some principal properties of circle homeomorphisms are inherited by the class of orientation-preserving circle maps. The latter class is rather broad and contains not only circle homeomorphisms but also a variety of non continuous maps arising in applications. Of course, even in cases when a property remains to be valid for orientation-preserving circle maps, absence of continuity sometimes results in noticeable changes of related proofs.

MSC 2000: 37E05, 37E10, 37E45

Key words: discontinuous map, orientation-preserving map, circle map, rotation number

1. INTRODUCTION

Orientation-preserving circle homeomorphisms possess a lot of interesting and non-trivial properties [4, 7] and play an important role in various fields of mathematics. Among such properties is the property of the rotation number of the homeomorphism f to be rational if and only if f has a periodic point, and also the Poincaré classification theorem giving conditions under which a circle homeomorphism is conjugate to a circle rotation map.

However, sometimes continuity of a map f may be restrictive (see, e.g., [1, Ch. VIII] or [2, 5]). Therefore, it is desirable to distinguish a class of circle maps into itself retaining as many as possible of the properties of homeomorphisms while remaining rather broad and containing not only circle homeomorphisms but also non continuous maps. One such class of maps will be considered below. It is the class of so-called orientation-preserving circle maps which in general are not continuous.

Of course, if a circle map lacks continuity than it inevitable loses some of its properties. An elementary examples in Section 3 demonstrate that a discontinuous circle map with rational rotation number may have no periodic points, or it may have periodic points with different coprime periods.

The paper is organized as follows. Sections 2–6 are devoted to the investigation of the case of single-valued discontinuous circle maps. In Section 2, basic properties of orientation-preserving circle maps and their lifts, strictly monotone maps of degree one, are discussed. Such maps are chosen in the paper as a replacement for circle homeomorphisms. Section 3 contains the definition of the rotation number $\tau(F)$ for the strictly monotone map $F : \mathbb{R} \to \mathbb{R}$ of degree one, and proofs of basic properties of $\tau(F)$ are also discussed. In Section 4, it is shown that $\tau(F)$ depends continuously on the graph of F in the Hausdorff semi-metric, which generalizes usual statements on

[1] The work was supported by grant No. 03-01-00258 of the Russian Foundation for Basic Research, by grant for Scientific Schools No. 1532.2003.1 of the President of Russian Federation. This work was partially written while the author was a visitor of both Boole Centre for Research in Informatics and Department of Applied Mathematics, University College Cork, financial support from which was gratefully acknowledged.

continuity of the rotation number of circle homeomorphism. In Section 5, it is proved that, in the case of irrational rotation number, iterations of a point under F are ordered like those for the corresponding rotation map. From this a restricted version of the Poincaré Classification Theorem for circle homeomorphisms is deduced, stating that an orientation-preserving circle map with irrational rotation number is semi-conjugate to a circle rotation map. Section 6 is devoted to investigation of the problem of whether or not an orientation-preserving circle map has bi-infinite trajectories. Finally, in Section 7 the case of set-valued circle maps with closed graphs is considered. A similar situation was investigated in [3] where the main results are established for the set-valued maps with connected images, whereas the set-valued maps studied in Section 7 may have disconnected images.

2. MONOTONE MAPS OF DEGREE ONE

Consider the class of all strictly monotone[1] maps $F : \mathbb{R} \to \mathbb{R}$ of degree one[2], i.e., class of all maps $F : \mathbb{R} \to \mathbb{R}$ satisfying

$$F(x+1) \equiv F(x) + 1, \qquad F(x) < F(y) \quad \text{for} \quad x < y. \tag{1}$$

Point out that generally maps satisfying (1) are not supposed to be continuous. At the same time namely continuous strictly monotone maps of degree one play an important role in investigation of circle homeomorphisms [4,7]. To be more precise, each strictly monotone continuous map $F : \mathbb{R} \to \mathbb{R}$ of degree one generates with the help of the relation

$$f(x) = F(x) \pmod 1 \tag{2}$$

the orientation-preserving homeomorphism f of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ which is convenient to treat as the interval $[0, 1)$ with topologically identified points 0 and 1. Reverse is also true: for any orientation-preserving circle homeomorphism f there exists infinitely many strictly monotone continuous maps $F : \mathbb{R} \to \mathbb{R}$ of degree one satisfying (2); such maps are called *lifts* of f. A strictly increasing lift F of the map f will be called *standard* if it satisfies $F(0) = f(0)$. It is worth pointing out here that any two lifts of the orientation-preserving circle homeomorphism f differ from each other on an integer constant.

Now, suppose that the map F is no longer continuous. What happens as a result of such supposition? This is the main question which will be studied below.

Notice first, that condition (1) implies

$$0 < F(y) - F(x) < 1 \quad \text{for} \quad 0 < y - x < 1. \tag{3}$$

From (1) and (3) the next lemma immediately follows.

Lemma 1. *Any iteration of strictly monotone map F of degree one is also strictly monotone map of degree one. The map $F_*(x) = F(x) - x$ is 1-periodic and satisfies*

$$|F_*(x) - F_*(y)| < 1, \qquad \forall\ x, y \in \mathbb{R}. \tag{4}$$

Mutual properties of maps $F : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{S}^1 \to \mathbb{S}^1$ tied by relation (2) are described by the following lemma (see, e.g., [6]).

[1] Throughout the paper the term *strictly monotone* is used as equivalent of the term *strictly increasing.*
[2] The map $F : \mathbb{R} \to \mathbb{R}$ are said to be *of degree* $k \in \mathbb{Z}$ if $F(x+1) \equiv F(x) + k$.

Lemma 2. *Let $F : \mathbb{R} \to \mathbb{R}$ be a strictly monotone map of degree one. Then for the map f defined by (2) there exist subintervals $I_+(f), I_-(f) \subseteq [0,1)$, one of which may be empty, such that*

(i) $0 \in I_+(f)$, $I_+(f) \cap I_-(f) = \emptyset$, $I_+(f) \cup I_-(f) = [0,1)$;

(ii) $f(x)$ is one-to-one increasing map on each of the intervals $I_+(f)$ and $I_-(f)$;

(iii) $f(x) > f(y)$ for any $x \in I_+(f)$, $y \in I_-(f)$

Conversely, for any map $f : \mathbb{S}^1 \to \mathbb{S}^1$ satisfying conditions (i)–(iii) there exists a strictly monotone lift $F : \mathbb{R} \to \mathbb{R}$ of degree one. Any two strictly monotone lifts of f of degree one differ from one another by a constant.

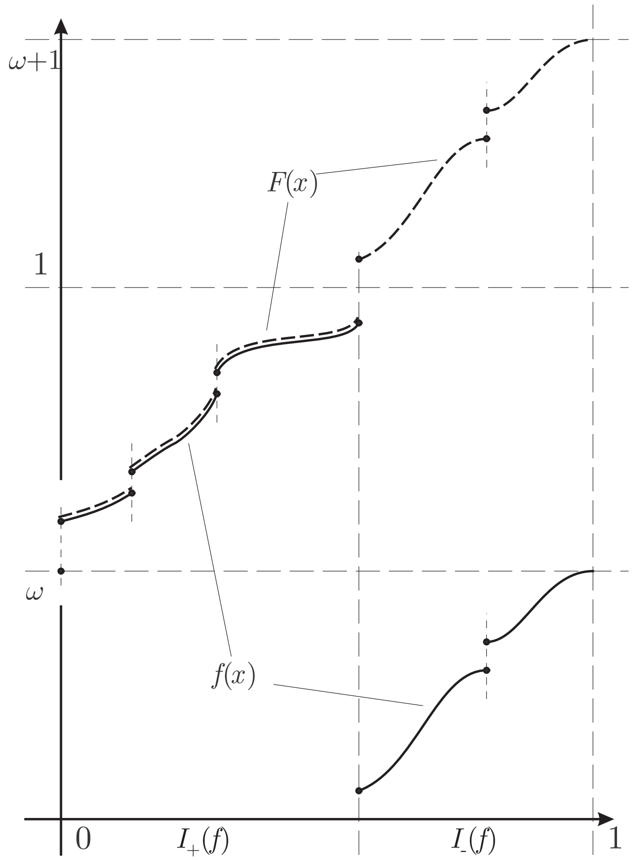

Figure 1. Orientation-preserving circle map $f(x)$ and its standard lift $F(x)$.

Maps $f : \mathbb{S}^1 \to \mathbb{S}^1$ satisfying conditions (i)–(iii) of Lemma 2 will be referred to as *orientation-preserving circle maps* (see, e.g., [6]). Typical plot of an orientation-preserving circle map is presented on Fig. 1. It is worth pointing out that under supposition that the map F is generally discontinuous, the corresponding orientation-preserving circle map f defined by (2) is also discontinuous.

3. ROTATION NUMBER

In this Section it will be shown that strictly monotone maps $F : \mathbb{R} \to \mathbb{R}$ of degree one share basic properties of lifts of circle homeomorphisms although proofs are changed comparing with traditional proofs which usually based on the continuity of related maps (see, e.g., [4, Ch. 11]).

Theorem 1. *Let $F:\mathbb{R}\to\mathbb{R}$ be a strictly monotone map of degree one. Then for any $x\in\mathbb{R}$ there exists independent from x number $\tau(F)$ (*the rotation number of the map F*) such that*

$$\left|\frac{F^n(x)-x}{n}-\tau(F)\right|\le\frac{2}{n}. \tag{5}$$

If the map $f=F\pmod 1$ has a q-periodic point then $\tau(F)$ is rational of the form p/q.

Proof. The proof is an insignificant modification of usual proofs known for the case of homeomorphisms (see, e.g., [4,7]) and is given below for the sake of completeness.

Fix an $x\in\mathbb{R}$ and an integer $n>0$ and set $F^{(n)}(x)=F^n(x)-x$. Then by Lemma 1

$$F^{(n)}(0)-1\le F^n(x)-x=F^{(n)}(x)\le F^{(n)}(0)+1. \tag{6}$$

Now, add together the relations (6) for points $x=y,F^n(y),\dots,F^{(m-1)n}(y)$ with an arbitrary $y\in\mathbb{R}$:

$$m(F^{(n)}(0)-1)\le F^{mn}(y)-y\le m(F^{(n)}(0)+1). \tag{7}$$

Dividing (7) by mn and subtracting from it the relation (6) divided by n, we get

$$\left|\frac{F^{mn}(y)-y}{mn}-\frac{F^n(x)-x}{n}\right|\le\frac{2}{n}. \tag{8}$$

Analogously can be obtained the relation

$$\left|\frac{F^{mn}(y)-y}{mn}-\frac{F^m(x)-x}{m}\right|\le\frac{2}{m}. \tag{9}$$

and thus,

$$\left|\frac{F^{m}(x)-x}{m}-\frac{F^n(x)-x}{n}\right|\le\frac{2}{n}+\frac{2}{m}. \tag{10}$$

From (10) it follows that $\{(F^n(x)-x)/n\}$ for any $x\in\mathbb{R}$ is a Cauchy sequence and so it has a limit $\tau(F,x)$. Then, firstly taking the limit in (8) as $m\to\infty$ we get

$$\left|\tau(F,y)-\frac{F^n(x)-x}{n}\right|\le\frac{2}{n}, \tag{11}$$

and secondly taking the limit in (11) as $n\to\infty$ we deduce that $|\tau(F,y)-\tau(F,x)|=0$, from which it follows that the limit $\tau(F,x)$ in fact does not depend on x, i.e., $\tau(F,x)\equiv\tau(F)$.

Now, from the identity $\tau(F,x)\equiv\tau(F)$ and (11) we obtain (5).

To finalize to proof it remained to show that the rotation number $\tau(F)$ is rational in the case when the map $f=\{F\}$ has a periodic point. Let $f^q(x)=x$ for some $x\in[0,1)$ and integer $q>0$. Then $F^q(x)=x+p$ for some integer p and therefore $F^{mq}(x)=x+mp$ for any integer $m=1,2,\dots$. Hence

$$\frac{F^{mq}(x)-x}{mn}=\frac{mp}{mn}=\frac{p}{q}$$

and taking the limit as $m\to\infty$ in the left side of the last equality we conclude that $\tau(F)=p/q$. Theorem is proved. □

If f is an orientation-preserving circle map and F is its strictly monotone lift of degree one then the value $\tau(f):=\tau(F)\pmod 1$ is called *the rotation number* of f. Since by Lemma 2 any two strictly monotone lifts of f of degree one differ from each other on an integer constant then the value $\tau(f)$ is well defined.

Remark 1. Unfortunately, the reverse statement, usual for homeomorphisms, that rationality of $\tau(F)$ implies the existence of a periodic point of the map $f = F \pmod 1$ is not valid under conditions of Theorem 1. Indeed, as is easy to see the map $f(x) = (x+1)/2$ defined on $[0, 1)$ has no periodic points while for any its strictly monotone lift F of degree one the equality $\tau(F) = 0$ is valid. Nevertheless, the corresponding statement is valid for discontinuous maps in a slightly modified form.

Given a strictly monotone map $F : \mathbb{R} \to \mathbb{R}$ of degree one, one can consider its *upper and lower associated maps*, F_+ and F_-, defined as

$$F_+(x) = \lim_{s\to x, s>x} F(x), \qquad F_-(x) = \lim_{s\to x, s<x} F(x).$$

Clearly, since $F(x)$ is monotone, maps F_+ and F_- are defined correctly and the both of them are strictly monotone maps of degree one satisfying

$$F_-(x) \le F(x) \le F_+(x). \tag{12}$$

Theorem 2. *Let $F : \mathbb{R} \to \mathbb{R}$ be a strictly monotone map of degree one with rational rotation number $\tau(F) = p/q$. Then either the map $f = F \pmod 1$ or the map $f_- = F_- \pmod 1$ or the map $f_+ = F_+ \pmod 1$ has a periodic point of period q.*

As it will be shown later in Theorem 4, $\tau(f) = \tau(f_-) = \tau(f_+)$. Then, by supposing in Theorem 4 that $\tau(f) = \tau(f_-)$ one may derive the following

Corollary 1. *Let $F : \mathbb{R} \to \mathbb{R}$ be a strictly monotone map of degree one with rational rotation number $\tau(F) = p/q$. Then either the map $f_- = F_- \pmod 1$ or the map $f_+ = F_+ \pmod 1$ has a periodic point of period q.*

To prove Theorem 2 we will need a simple fixed-point statement concerning monotonic maps.

Lemma 3. *Let $h : [a, b] \to \mathbb{R}^1$ with $-\infty < a < b < \infty$ be a non-decreasing map.*[3] *If $h(a) \ge a$ and $h(b) \le b$ then there exists such an $x_* \in [a, b]$ for which $h(x_*) = x_*$.*

Proof. If $h(a) = a$ or $h(b) = b$ then Lemma is proved. So, without loss in generality one may suppose that $h(a) > a$ and $h(b) < b$. Consider the set

$$X_* = \{x : \ h(x) > x, x \in [a, b]\}.$$

From supposition that $h(a) > a$ and $h(b) < b$ it follows that

$$[a, h(a)) \subseteq X_*, \qquad (h(b), b] \subseteq [a, b]\backslash X_*, \tag{13}$$

since by monotonicity of the function h one have:

$$x < h(a) \le h(x), \qquad \text{for} \quad x \in [a, h(a))$$

and

$$h(x) \le h(b) < x, \qquad \text{for} \quad x \in (h(b), b].$$

From (13) it follows that the set X_* contains infinitely many points and thus possesses a maximal accumulation point x_*, i.e. such a point that in each its left neighborhood there are infinitely many

[3] The map h is not supposed to be continuous.

points from X_* and to the right from it there are only finitely many points from X_*. Then to the right from x_* there are infinitely many points from $[a,b]\backslash X_*$. Hence there exist $y_n \to x_*$ and $z_n \to x_*$, such that

$$y_n < x_*, \quad y_n < h(y_n) \qquad \text{and} \qquad x_* \le z_n, \quad h(z_n) \le z_n \tag{14}$$

for all $n = 1, 2, \ldots$.

From (14) it follows that

$$y_n < h(y_n) < h(x_*) \le h(z_n) \le z_n$$

and taking here the limit when $y_n \to x_*$ and $z_n \to x_*$ we get $h(x_*) = x_*$. □

In proving Theorem 2 we will follow the scheme of proof of the corresponding statement from [4, see. Prop. 11.1.4] with necessary changes caused by possible discontinuity of the map F.

Proof of Theorem 2. By definition of the rotation number $\tau(f) = \tau(F) \pmod 1$ we have

$$\tau(f^q) = \lim_{n\to\infty} \frac{1}{n}\left((F^q)^n(x) - x)\right) = q \lim_{n\to\infty} \frac{1}{qn}\left((F^{qn}(x) - x)\right) = q\tau(f) \pmod 1.$$

So, $\tau(f^q) = 0$ since the rotation number of the map f is defined with the accuracy to an integer. Then to prove Theorem it suffices to show that the relation $\tau(f) = 0$ implies that either f_- or f_+ has a fixed point.

Consider now such a lift F of the map f for which $F(0) \in [0,1)$. If $F(x) - x \le 0$ for some $x \in [0,1)$ then by Lemma 3 the map F has a fixed point which implies that the map f also has a fixed point. Analogously, if $F(x) - x \ge 1$ for some $x \in [0,1)$ then by Lemma 3 the map $F - 1$ has a fixed point from which again follows the existence of a fixed point for the map f. So, we should only consider the case when

$$0 < F(x) - x < 1 \quad \text{for} \quad x \in [0,1).$$

If

$$\inf_{0\le x<1} \{F(x) - x\} = 0$$

then either $\min_{0\le x\le 1}\{F_-(x) - x\} = 0$ or $\min_{0\le x\le 1}\{F_+(x) - x\} = 0$. In the former case the map F_- has a fixed point while in the latter case the map F_+ has a fixed point, and in both cases Theorem is proved.

If

$$\sup_{0\le x<1} \{F(x) - x\} = 1$$

then either $\max_{0\le x\le 1}\{F_-(x) - x\} = 1$ or $\max_{0\le x\le 1}\{F_+(x) - x\} = 1$. In the former case the map $F_- - 1$ has a fixed point while in the latter case the map $F_+ - 1$ has a fixed point. This means that either the map f_- or the map f_+ has a fixed point. So, again, in both cases Theorem is proved.

It remained to consider only the case, when there exists such a $\delta > 0$ for which

$$\delta < F(x) - x < 1 - \delta \quad \text{for} \quad x \in [0,1).$$

Putting in the above inequalities the values $x = F^i(0)$ and sum the resulting estimates from $i = 0$ to $i = n-1$ we get

$$n\delta < F^n(0) < n(1-\delta)$$

or

$$\delta < \frac{F^n(0)}{n} < 1 - \delta.$$

Now, taking here the limit as $n \to \infty$ we conclude that $\delta < \tau(F) < 1 - \delta$ and thus $\tau(f) \ne 0$. A contradiction, which completes the proof of Theorem. □

4. CONTINUITY OF THE ROTATION NUMBER IN THE HAUSDORFF METRIC

As is known, the rotation number $\tau(f)$ of a circle homeomorphism f depends continuously on f in the topology of uniform convergence (see, e.g., [4, Prop. 11.1.6]). Clearly, the same is valid for rotation numbers of strictly monotone continuous maps of $\mathbb{R}$ of degree one. In the general case, when considering discontinuous maps, the uniform or even pointwise convergence is too restrictive. So, below it will be proposed a more general result on continuity of the function $\tau(F)$.

Denote by $\Gamma(F) := \{z \in \mathbb{R}^2 : \ z = (F(x), x), x \in \mathbb{R}\}$ the graph of the map F. Denote by $\|z\|$ the max-norm in $\mathbb{R}^2$, i.e., $\|z\| = \max\{|z_1|, |z_2|\}$. And, at last, define the Hausdorff semi-metric between graphs of strictly monotone maps F and G of degree one as

$$\chi(F,G) = \max\left\{ \sup_{z\in\Gamma(F)} \inf_{u\in\Gamma(G)} \|z-u\|, \quad \sup_{u\in\Gamma(G)} \inf_{z\in\Gamma(F)} \|u-z\| \right\}.$$

Point out that $\chi(F,G)$ possesses all the properties of metric except one: since graph of discontinuous map is not closed then it may happen that $\chi(F,G) = 0$ while $F \neq G$. Generally, convergence defined by the semi-metric $\chi(F,G)$ is weaker than uniform or even pointwise convergence. Nevertheless, there are situations when χ-convergence implies pointwise convergence.

Lemma 4. *Let m be an integer and let $x, F(x), \ldots, F^{m-1}(x)$ be points of continuity for the map F and $\chi(F, F_n) \to 0$. Then $F_n^m(x_n) \to F^m(x)$ for any sequence $\{x_n\}$ such that $x_n \to x$.*

Proof. Prove first Lemma for the case $m = 1$. Given the sequences $\{F_n\}$ and $\{x_n\}$, by definition of the Hausdorff metric χ for any $n = 1, 2, \ldots$ it may be chosen $y_n = (F(z_n), z_n) \in \gamma(F)$ such that

$$\|(F_n(x_x), x_n) - y_n\| \leq \chi(F, F_n).$$

Then by definition of the max-norm $\|\cdot\|$

$$|x_n - z_n| \leq \chi(F, F_n) \to 0, \tag{15}$$

$$|F_n(x_n) - F(z_n)| \leq \chi(F, F_n) \to 0. \tag{16}$$

From (15) and condition that $x_n \to x$ it follows that $z_n \to x$. Then by continuity of the map F at the point x we get $F(z_n) \to F(x)$ and in view of (16) $F_n(x_n) \to F(x)$. Lemma is proved for the case $m = 1$.

Ie the general case Lemma can be proved by induction. Suppose that the statement of lemma is valid for $k = p - 1$ with $1 \leq p - 1 < m$, prove that then it is valid for $k = p$.

By supposition $u_n = F^{p-1}(x_n) \to F^{p-1}(x)$ as $x_n \to x$ where by condition of Lemma $F^{p-1}(x)$ is the point of continuity of F. Then by the already proven statement of Lemma for the case $m = 1$ we get $F^p(x_n) = F(u_n) \to F(F^{p-1}(x)) = F^p(x)$. The step of induction is completed and so, Lemma is proved. □

Theorem 3. *Let F, F_n, $n = 1, 2, \ldots$, be strictly monotone maps of degree one such that $\chi(F, F_n) \to 0$ as $n \to \infty$. Then $\tau(F_n) \to \tau(F)$ as $n \to \infty$.*

Proof. Denote by $\mathbb{D}_1(F)$ the set of all points of discontinuity for the map F; since F by supposition is monotone then the set $\mathbb{D}_1(F)$ is countable. By supposition the map F is not only monotone, it is strictly monotone and thus injective. Then the set $\mathbb{D}_2(F) := \{x : \ F(x) \in \mathbb{D}_1(F)\}$ is also countable. Analogously, each set $\mathbb{D}_n(F) := \{x :\ F^n(x) \in \mathbb{D}_1(F)\}$, $n = 2, 3, \ldots$, is also countable[4]. Then the set

$$\mathbb{D}(F) = \bigcup_{n \geq 1} \mathbb{D}_n(F).$$

is also countable. Hence the set $\mathbb{C}(F) = \mathbb{R} \backslash \mathbb{D}(F)$ consisting of all $x \in \mathbb{R}$ such that $x, F(x), F^2(x), \ldots$ are points of continuity for the map F is not empty.

Choose now an $\varepsilon > 0$ an fix some $x \in \mathbb{C}(F)$. Then by Theorem 1 for any integer m satisfying $m \geq 6/\varepsilon$ there will be valid estimate

$$\left| \frac{F^m(x) - x}{m} - \tau(F) \right| \leq \frac{\varepsilon}{3}. \tag{17}$$

Fix any m for which the above estimate is true. Then, by definition of the set $\mathbb{C}(F)$ and choice of the point $x \in \mathbb{C}(F)$, according to Lemma 4 $F_n^m(x) \to F^m(x)$ as $n \to \infty$. Hence such an $N(\varepsilon)$ can be chosen that

$$\left| \frac{F_n^m(x) - x}{m} - \frac{F^m(x) - x}{m} \right| \leq \frac{\varepsilon}{3} \qquad \text{as} \quad n \geq N(\varepsilon). \tag{18}$$

At last, again by Theorem 1 since $m \geq 6/\varepsilon$ then

$$\left| \frac{F_n^m(x) - x}{m} - \tau(F_n) \right| \leq \frac{2}{m} \leq \frac{\varepsilon}{3}, \qquad \forall\ n. \tag{19}$$

From (17), (18) and (19) one can deduce that $|\tau(F_n) - \tau(F)| \leq \varepsilon$ for $n \geq N(\varepsilon)$ and hence $\tau(F_n) \to \tau(F)$ as $n \to \infty$. Theorem is proved. □

Now, one important corollary of Theorem 3 specific to discontinuous strictly monotone maps of degree one will be proved. Strictly monotone maps F and G of degree one will be called *equivalent* if

$$F_-(x) \leq G(x) \leq F_+(x), \qquad x \in \mathbb{R}. \tag{20}$$

Clearly, relations $F_-(x) \leq G(x) \leq F_+(x)$ imply relations $G_-(x) \leq F(x) \leq G_+(x)$, so the definition of equivalency of F and G is correct.

Theorem 4. *If F and G are equivalent strictly monotone maps of degree one then $\tau(F) = \tau(G)$.*

Proof. From definition of the rotation number it follows that $\tau(F_1) \leq \tau(F_2)$ if $F_1(x) \leq F_2(x)$ for $x \in \mathbb{R}$. Then the relations

$$F_-(x) \leq F(x), G(x) \leq F_+(x)$$

imply

$$\tau(F_-) \leq \tau(F), \tau(G) \leq \tau(F_+). \tag{21}$$

Now, from the fact that $\overline{\Gamma(F_-)} = \overline{\Gamma(F_+)}$[5] the relation $\chi(F_-, F_+) = 0$ follows. Then by Theorem 3 $\tau(F_-) = \tau(F_+)$ which, in view of (21), implies that $\tau(F) = \tau(G)$. Theorem is proved. □

[4] Strictly speaking, each of the sets $\mathbb{D}_n(F)$ consists of no more that countably many points.

[5] Remark, that generally $\chi(F, F_+) \neq 0$ and $\chi(F, F_-) \neq 0$ as is, for example, in the case when $F(x_0) \neq F_-(x_0)$ and $F(x_0) \neq F_+(x_0)$ for some x_0. Clearly, x_0 in this case is such a point of discontinuity of F for which $(F(x_0, x_0))$ is an isolated point of the graph $\Gamma(F)$.

5. SEMI-CONGUJACY WITH A CIRCLE SHIFT MAP

One of the most important results of the theory of circle homeomorphisms is one stating that each circle homeomorphisms with irrational rotation number semi-conjugate to a circle shift (or rotation) map

$$\rho_\tau(x) := x + \tau \pmod 1, \qquad x \in [0,1). \tag{22}$$

As it turned out the same result is valid also for generally discontinuous orientation-preserving circle maps. It is worth pointing out that generally related proofs are changed.

Prove first that orbits of an orientation-preserving circle map f with irrational rotation number $\tau(f)$ are ordered exactly as those for the circle shift map ρ_τ with $\tau = \tau(f)$.

Lemma 5. *Let F be a strictly monotone lift of degree one of an orientation-preserving circle map f with irrational rotation number $\tau = \tau(F)$. Then for any $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ and $x \in \mathbb{R}$*

$$n_1\tau + m_1 < n_2\tau + m_2 \quad \textit{if and only if} \quad F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2.$$

Proof. First consider the case when $F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2$ and $n_1 < n_2$. By setting $y = F^{n_1}(x)$ the former inequality is equivalent to $y < F^{n_2-n_1}(y) + m_2 - m_1$. From this, since the map F is strictly monotone and of degree one, we obtain

$$y < F^{n_2-n_1}(y) + m_2 - m_1 < F^{n_2-n_1}(F^{n_2-n_1}(y) + m_2 - m_1) + m_2 - m_1 = \\ = F^{2(n_2-n_1)}(y) + 2(m_2 - m_1).$$

Inductively,

$$y < F^{k(n_2-n_1)}(y) + k(m_2 - m_1), \qquad k = 1, 2, \ldots,$$

and so

$$\tau = \tau(F) = \lim_{k\to\infty} \frac{F^{k(n_2-n_1)}(y) - y}{k(n_2 - n_1)} > \lim_{k\to\infty} \frac{k(m_2 - m_1)}{k(n_2 - n_1)} = \frac{m_2 - m_1}{n_2 - n_1}$$

(with a strict inequality due to irrationality of τ). Hence,

$$n_1\tau + m_1 < n_2\tau + m_2.$$

Now, consider the case when $F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2$ while $n_1 > n_2$. Then by setting $y = F^{n_2}(x)$ we get $F^{n_1-n_2}(y) + m_1 - m_2 < y$. From this, as in the previous case, we obtain

$$F^{k(n_1-n_2)}(y) + k(m_1 - m_2) < y, \qquad k = 1, 2, \ldots,$$

Then

$$\tau = \tau(F) = \lim_{k\to\infty} \frac{F^{k(n_1-n_2)}(y) - y}{k(n_1 - n_2)} < \lim_{k\to\infty} \frac{k(m_1 - m_2)}{k(n_1 - n_2)} = \frac{m_1 - m_2}{n_1 - n_2}$$

(with a strict inequality due to irrationality of τ) which again imply

$$n_1\tau + m_1 < n_2\tau + m_2.$$

Thus we have proved that $F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2$ implies $n_1\tau + m_1 < n_2\tau + m_2$. Similarly $F^{n_1}(x) + m_1 > F^{n_2}(x) + m_2$ implies $n_1\tau + m_1 > n_2\tau + m_2$ and equality in the considered relations never occurs (since τ is irrational and thus F has no periodic points). So, the lemma is proved. $\square$

The preceding lemma demonstrates that in the case of irrational rotation number iterations of a point under F ordered like those for the corresponding rotation. The following Theorem is a restricted version of the Poincaré Classification Theorem for circle homeomorphisms [4, Th. 11.2.7].

Theorem 5. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving map (generally discontinuous) with irrational rotation number $\tau = \tau(f)$. Then map $\rho_\tau(x) := x + \tau \pmod 1$ is a topological factor*[6] *of f via continuous orientation-preserving map $h : \mathbb{S}^1 \to \mathbb{S}^1$.*

Proof. Let F be a strictly monotone lift of degree one of the map f (such a lift exists due to Lemma 2). Consider for an arbitrary $x \in \mathbb{R}$ the set

$$B = B(x) := \{F^n(x) + m : \quad n, m \in \mathbb{Z}\}$$

and define the map

$$H : B \to \mathbb{R} \quad \text{such that} \quad F^n(x) + m \mapsto n\tau + m$$

where $\tau = \tau(F)$. Then by Lemma 5 the map H is monotone (moreover, it is a map of degree one). Note also that due to irrationality of τ the set $H(B)$ is dense in $\mathbb{R}$. So, if we use the notation R_τ for the map $R_\tau : x \mapsto x + \tau$, then $H \circ F = R_\tau \circ H$ since

$$H \circ F(F^n(n) + m) = H(F^{n+1}(x) + m) = (n+1)\tau + m$$

and

$$R_\tau \circ H(F^n(x) + m) = R_\tau(n\tau + m) = (n+1)\tau + m.$$

Prove now that H has a continuous extension to the closure $\bar{B}$ of B. Indeed, if $y \in \bar{B}$ then there exists a sequence $\{y_n\} \subset B$ such that $y = \lim_{n\to\infty} y_n$. To define by continuity H at the point y we should set $H(y) := \lim_{n\to\infty} H(y_n)$. To show that $\lim_{n\to\infty} H(y_n)$ exists and does not depend on the choice of the sequence approximating y observe first that the left and right limits exist and are independent of the sequence since H is monotone. At last, note that the left and right limits will coincide as in the opposite case the set $\mathbb{R} \setminus H(B)$ contains an interval. So, we have proved that H has a continuous extension to the closure $\bar{B}$ of B.

Now, H can easily be extended to $\mathbb{R}$. Since $H : \bar{B} \to \mathbb{R}$ is monotone and surjective (since H is monotone and continuous on B, $\bar{B}$ is closed, and $H(B)$ is dense in $\mathbb{R}$) there is no choice in defining H on the intervals complementary to $\bar{B}$ as to set $H = \text{const}$ on those intervals, choosing the constant value equal to the values at endpoints. This defines the map $H : \mathbb{R} \to \mathbb{R}$ satisfying $H \circ F = R_\tau \circ H$ which is of degree one since for $y = F^n(x) + m \in B$ we have

$$H(y+1) = H(F^n(x) + m + 1) = n\tau + m + 1 = H(y) + 1$$

and this property persists under continuous extension.

Now, from $H \circ F = R_\tau \circ H$ it follows that $h \circ f = \rho_\tau \circ h$ with $h(x) = H(x) \pmod 1$ and $\rho_\tau(x) = R_\tau(x) \pmod 1 \equiv x + \tau \pmod 1$. □

Corollary 2. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving map with irrational rotation number and let $I \subset \mathbb{S}^1$ be a closed interval with endpoints $f^m(x)$ and $f^n(x)$ where $m \neq n$ are positive integers. Then for any $y \in \mathbb{S}^1$ there is a positive integer k such that $f^k(y) \in I$.*[7]

Proof. The conjugating map h constructed in the proof of Theorem 5 maps the points $f^m(x)$ and $f^n(x)$ to the points $\varphi_1 = m\tau \pmod 1$ and $\varphi_2 = n\tau \pmod 1$ respectively. Since τ is irrational and $m \neq n$ then $\varphi_1 \neq \varphi_2$. Then, again by irrationality of τ, for any $y \in \mathbb{S}^1$ there exists a positive integer k such that $h(f^k(y)) = h(y) + k\tau \pmod 1 \in [\varphi_1, \varphi_2]$. From this, since h is monotone and continuous, we get that $f^k(y) \in h^{-1}([\varphi_1, \varphi_2]) = I$. □

Corollary 3. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving map with irrational rotation number. Then the ω-limit set*[8] *$\omega(x)$ is independent of x.*

[6] Remind, that a map $g : Y \to Y$ is a topological factor of the map $f : X \to X$ if there exists a surjective continuous map $h : X \to Y$ such that $h \circ f = g \circ h$.

[7] There are exactly two intervals in $\mathbb{S}^1$ with endpoints $f^m(x)$ and $f^n(x)$; the corollary is valid for either case.

[8] The ω-limit set for a point x is defined as the set of all limiting points of the sequence $\{f^n(x)\}_{n=1}^{\infty}$.

Proof. We need to show that $\omega(x) = \omega(y)$ for $x, y \in \mathbb{S}^1$. Let $z \in \omega(x)$. Then there is a sequence $m_n > 0$ such that $f^{m_n}(x) \to z$. By Corollary 2 for $y \in \mathbb{S}^1$ there exist $k_n > 0$ such that $f^{k_n}(y) \in [f^{m_n}(x), f^{m_{n+1}}(x)]$. Thus $\lim_{n\to\infty} f^{k_n}(y) = \lim_{n\to\infty} f^{m_n}(x) = z$. Therefore $\omega(y) \subseteq \omega(x)$ for all $, y \in \mathbb{S}^1$ and by symmetry $\omega(y) = \omega(x)$ for all $x, y \in \mathbb{S}^1$. □

Clearly, any ω-limit set is closed. In the case when f is a circle homeomorphism the set $\omega = \omega(x)$ is also invariant with respect to f, i.e., $f(\omega) = \omega$, while for an orientation-preserving circle map we can not even state that $f(\omega) \subseteq \omega$.

6. BI-INFINITE TRAJECTORIES

One of the most important features of circle homeomorphisms is that for any $x \in \mathbb{S}^1$ there exists a bi-infinite trajectory $\{x_n\}_{n=-\infty}^{\infty}$ of the corresponding map f satisfying $x_0 = x$, i.e.,

$$x_{n+1} = f(x_n), \quad -\infty < n < \infty, \qquad x_0 = x. \tag{23}$$

Clearly, orientation-preserving circle maps generally do not possess the above feature as for them the image $f(\mathbb{S}^1)$ may be a proper part of $\mathbb{S}^1$ and so there may exist points with no preimages at all. Nevertheless, as is stated by Theorem 6 below in under some conditions the set $\omega_\infty(f)$ of all points $x \in \mathbb{S}^1$ for which there exists a bi-infinite trajectory $\{x_n\}_{n=-\infty}^{\infty}$ hitting x at zero time (see (23)) is not empty.

Theorem 6. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving map with irrational rotation number $\tau(f)$. Then $\omega_\infty(f) \neq \emptyset$.*

To prove Theorem 6 we need two auxiliary statements. First we shall prove that Theorem 6 is valid under supposition that the map f is semi-continuous (from the left or from the right). Then we shall prove that for semi-continuous maps the set ω_∞ in not only non-empty; the cardinality of this set is continuum. From this we shall deduce that analogous properties are valid for general maps satisfying conditions of Theorem 6.

Lemma 6. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving semi-continuous from the left (from the right) map with irrational rotation number $\tau = \tau(f)$, and $x \in \mathbb{S}^1$. Then any limiting point of a monotone subsequence[9] $f^{n_0}(x) < f^{n_1}(x) < \ldots < f^{n_k}(x)\ldots$ belongs to the set $\omega_\infty(f)$ and so $\omega_\infty(f) \neq \emptyset$.*

Proof. Given an $x \in \mathbb{S}^1$, prove first that there exists at least one bounded increasing sequence of the form $\{f^{n_k}(x)\}$. Fix some positive integers m, n such that $0 \leq f^n(x) < f^m(x) < 1$. By Corollary 2 there is a number n_0 such that $f^n(x) \leq f^{n_0}(x) \leq f^m(x) < 1$. Then, again by Corollary 2 there is a number $n_1 > n_0$ such that $f^{n_0}(x) \leq f^{n_1}(x) \leq f^m(x) < 1$, etc. Hence there is a sequence of positive integers $\{n_k\}$ such that

$$0 \leq f^{n_0}(x) \leq f^{n_1}(x) \leq \ldots \leq f^m(x) < 0. \tag{24}$$

Notice, that all the inequalities in (24) are, in fact, strict since due to irrationality of $\tau(f)$ the map f has no periodic points.

So, the existence of bounded increasing sequences $\{f^{n_k}(x)\}$ is proved. Let $\{f^{n_k}(x)\}$ be one of such sequences, then denote $z_0 = \lim_{k\to\infty} f^{n_k}(x)$. Show that $z_0 \in \omega_\infty(f)$ and thus $\omega_\infty(f) \neq \emptyset$. Consider the sequence $\{f^{n_k-1}(x)\}$. Since this is a sequence from $\mathbb{S}^1$ then it is compact and without loss in generality it may be treated as converging, i.e., there exists $z_1 \in \mathbb{S}^1$ such that $f^{n_k-1}(x) \to z_1$ as $k \to \infty$. But in view of local monotonicity of the map f (see Lemma 2) the sequence $\{f^{n_k-1}(x)\}$ should

[9] Here to use the monotonicity arguments we identify $\mathbb{S}^1$ with $[0, 1)$.

be non-decreasing since the sequence $\{f^{n_k}(x)\}$ is non-decreasing by definition. Then $f^{n_k-1}(x) \le z_1$ and

$$f(z_1) = \lim_{k\to\infty} f(f^{b_k}(x)) \equiv \lim_{k\to\infty} f^{n_k}(x) = z_0$$

where the first limit is valid due to supposition that $f(x)$ is semi-continuous from the left. So, $f(z_1) = z_0$. Analogously, there exists $z_2 \in \mathbb{S}^1$ such that $f(z_2) = z_1$, etc.

From the above reasoning it follows the existence of sequence $\{z_k\}$ such that $f(z_{k+1}) = z_k$, $k = 0, 1, \ldots$, which means that $z_0 \in \omega_\infty(f)$ and thus $\omega_\infty(f) \ne \emptyset$. □

Clearly, by definition $f(\omega_\infty) = \omega_\infty$. From the proof of Lemma 6 it is also seen that $\omega_\infty(f) = \omega(f)$ for the semi-continuous from the left of from the right map f with irrational rotation number.

Lemma 7. *Let f satisfy conditions of Lemma 6. Then for any $x \in \mathbb{S}^1$ the cardinality of the set of limiting points of all growing sequences $f^{n_0}(x) < f^{n_1}(x) < \ldots < f^{n_k}(x)\ldots$ is continuum. So the cardinality of $\omega_\infty(f)$ is also continuum.*

Proof. Given an $x \in \mathbb{S}^1$ fix positive integers n_1, n_2, n_3, n_4 such that

$$0 < f^{n_1}(x) < f^{n_2}(x) < f^{n_3}(x) < f^{n_4}(x) < 1;$$

such integers exist by Lemma 5 since, by supposition, $\tau(f)$ is irrational and so all the points $f^k(x)$, $k = 0, 1, \ldots$, are distinctive. Define intervals of "zero level"

$$\Delta_0 = [f^{n_1}(x), f^{n_2}(x)], \qquad \Delta_1 = [f^{n_3}(x), f^{n_4}(x)].$$

Then choose inside interval Δ_0 four points $f^{n_{01}}(x), f^{n_{02}}(x), f^{n_{03}}(x), f^{n_{04}}(x)$ satisfying

$$f^{n_1}(x) < f^{n_{01}}(x) < f^{n_{02}}(x) < f^{n_{03}}(x) < f^{n_{04}}(x) < f^{n_1}(x)$$

and define intervals of the "first level"

$$\Delta_{00} = [f^{n_{01}}(x), f^{n_{02}}(x)] \subset \Delta_0, \qquad \Delta_{01} = [f^{n_{03}}(x), f^{n_{04}}(x)] \subset \Delta_0.$$

Analogously, we can choose inside interval Δ_1 four points $f^{n_{11}}(x)$, $f^{n_{12}}(x)$, $f^{n_{13}}(x)$ and $f^{n_{14}}(x)$ satisfying

$$f^{n_3}(x) < f^{n_{11}}(x) < f^{n_{12}}(x) < f^{n_{13}}(x) < f^{n_{14}}(x) < f^{n_4}(x)$$

and define two more intervals of the "first level"

$$\Delta_{10} = [f^{n_{11}}(x), f^{n_{12}}(x)] \subset \Delta_1, \qquad \Delta_{11} = [f^{n_{13}}(x), f^{n_{14}}(x)] \subset \Delta_1.$$

The procedure of construction of the Δ-intervals can be continued by induction. Provided that we have got already 2^{n+1} intervals of the 2^nth level, we can choose in each of such intervals 2 new intervals with endpoints from the set $\{f^k(x)\}_{k=1}^\infty$ in such a way that the endpoints of all the Δ-intervals (old and newborn) would be distinctive.

So, such a procedure results in construction of a set of intervals with distinctive endpoints taken from the set $\{f^k(x)\}_{k=1}^\infty$, subdivided in "levels". On the highest, zero level there are two such intervals. On the n-th level there are 2^{n+1} intervals, and each of them contains exactly 2 intervals from the next $(n + 1)$-th level.

As is easy to see this procedure resembles the construction of a Cantor set. The only difference is that, due to the fact that endpoints of our intervals are distinctive, the intersection of any infinite filtered sequence of such intervals[10] is non-empty and has no common points with another

[10] The sequence of intervals $\{\Delta_n\}$ is called *filtered* if $\Delta_0 \supseteq \Delta_1 \supseteq \ldots \supseteq \Delta_n \supseteq \ldots$.

such interval determined by a different filtered sequence of intervals. Hence, the unity of all the intersections of all the filtered sequences from our set of intervals has cardinality of all the binary sequences which is continuum.

Note now, that for any filtered sequence of intervals $\{\Delta_k\}$ from our set of intervals their left end-points increase and have the form $f^{n_k}(x)$. So, each filtered sequence of intervals uniquely determines the point

$$z = \lim_{k\to\infty} f^{n_k}(x), \qquad f^{n_k}(x) < f^{n_{k+1}}(x) < z, \quad k = 1, 2, \ldots, \tag{25}$$

and cardinality of different points defined in such a manner is continuum.

At last, by Lemma 6 any point defined by (25) belongs to $\omega_\infty(f)$, so the cardinality of $\omega_\infty(f)$ is also continuum. □

Proof of Theorem 6. Define an auxiliary map

$$\tilde{f}(x) := \lim_{y\to x, y<x} f(x).$$

Then $\tilde{f}(x)$ is a semi-continuous from the left orientation-preserving circle map. Since $f(x)$ and $\tilde{f}(x)$ may differ only at points of discontinuity of $f(s)$ while $f(x)$ has only countably many points of discontinuity then the set

$$D_f := \{x \in \mathbb{S}^1 : \quad f(x) \neq \tilde{f}(x)\}$$

is finite or countable. Therefore the set

$$D_f^\infty : \{x \in \mathbb{S}^1 : \quad f^n(x) \in D_f \text{ for some integer} n \geq 0\}$$

is also finite or countable due to injectivity of the map f. Hence the set $\mathbb{S}^1 \setminus D_f^\infty$ is not empty.

Choose an $x \in \mathbb{S}^1 \setminus D_f^\infty$. By definition of the set $\mathbb{S}^1 \setminus D_f^\infty$ all the points $f^n(x)$, $n = 0, 1, \ldots$, are points of continuity of the map $f(x)$ and therefore

$$f^n(x) = \tilde{f}^n(x), \qquad n = 0, 1, \ldots. \tag{26}$$

By Lemmas 6 and 7 the cardinality of the set $\omega_\infty(\tilde{f})$ is continuum. Moreover, by Lemma 7 the set $\omega_{\tilde{f}}$ contains continuum of points which are limits from the left of increasing subsequences of the form $\tilde{f}^{n_k}(x)$. Since the set D_f^∞ is countable, then there exists an increasing subsequence $\tilde{f}^{n_k}(x)$ converging to a point $z \notin D_f^\infty$. In this case by Lemma 6 $z \in \omega_\infty(\tilde{f})$ but since $z \notin D_f^\infty$ then in fact $z \in \omega_\infty(f)$. So, $\omega_\infty(f) \neq \emptyset$. □

7. SET-VALUED CIRCLE MAPS

Recall basic facts of the theory of set-valued orientation-preserving discontinuous circle maps, following primarily to the work [3]. Point out that the basic definitions and constructions developed in the previous Sections for single-valued circle maps can be applied without changes to set-valued circle maps. Nevertheless, to avoid misunderstanding let us present necessary definitions and facts.

Let $f : [0, 1) \to [0, 1)$ be some, in general, discontinuous, set-valued function. The function $F : \mathbb{R} \to \mathbb{R}$ is called *the lift* of f if it satisfies conditions

$$F(x + 1) \equiv F(x) + 1, \tag{27}$$

and

$$f(x) = F(x) \pmod 1 \qquad x \in [0, 1). \tag{28}$$

Each set-valued circle map has a lift, and conversely, each map F of the straight line in itself satisfying (27) is a lift of the circle map $f(\cdot)$ defined by the equality (28).

The set-valued map $f : [0,1) \to [0,1)$, treated as a map of the circle $\mathbb{S} \equiv \mathbb{R}/\mathbb{Z}$ in itself, will be called *orientation-preserving* if it has a strictly increasing lift. The orientation-preserving map $f : [0,1) \to [0,1)$ will be called *closed* or *connectedly closed* if it has a strictly increasing lift with the closed graph, or the graph of some of its strictly increasing lift is a connected and closed set, respectively.

To illustrate notions introduced above, associate with the strictly increasing lift F of the map f the auxiliary maps

$$F_+(x) = \lim_{\bar{x}\downarrow x} F(\bar{x}), \qquad F_-(x) = \lim_{\bar{x}\uparrow x} F(\bar{x}),$$

where notations $\bar{x} \downarrow x$ and $\bar{x} \uparrow x$ are used to denote convergence of the variable $\bar{x}$ to x strictly from above or from below, correspondingly. Define also the maps

$$F_*(x) = \{F_-(x), F_+(x)\}, \qquad F_c(x) = [F_-(x), F_+(x)].$$

Directly from the definitions of the maps $F_+(x)$, $F_-(x)$, $F_*(x)$ and $F_c(x)$ it follows that all these maps are strictly increasing. The maps $F_+(x)$ and $F_-(x)$ are single-valued, and the map $F_+(x)$ is continuous from the right at each point, while the map $F_-(x)$ is continuous from the left at each point. The maps $F_*(x)$ and $F_c(x)$ are, in general, set-valued and their values coincide with the values of the map $F(x)$ at the points, in which the map $F(x)$ is single-valued and continuous. In all other points the values of $F_*(x)$ consist of exactly two points while the values of $F_c(x)$ consist of closed intervals. Besides, the graphs of the both maps $F_*(x)$ and $F_c(x)$ are closed. It should be noted also that

$$F_+(x), F_-(x) \in F_*(x) \subseteq F_c(x) \qquad \forall x.$$

In addition, if the graph of the map $F(x)$ is closed then $F_*(x) \subseteq F(x) \subseteq F_c(x)$. Therefore, it is natural to call the map $F_*(x)$ the *minimal closure* of the map $F(x)$ while the map $F_c(x)$ can be called the *connected* or *maximal closure* of the map $F(x)$. Respectively, the map $F(x)$ will be called *minimally closed* if $F(\cdot) = F_*(\cdot)$, and it will be called *connectedly* or *maximally closed* if $F(\cdot) = F_c(\cdot)$.

Theorem 7 (see [3]). *Let $f : [0,1) \to [0,1)$ be an orientation-preserving circle map with a connectedly closed lift F. Let $\{x_n\}$ be a trajectory of the map F, i.e.*

$$x_{n+1} \in F(x_n), \qquad n = 0, 1, \dots . \tag{29}$$

Then the following assertions are valid:

(i) there is a number τ, not depending on the initial value x_0, for which the estimates hold

$$\left|\frac{x_n}{n} - \tau\right| \leq \frac{1}{n}, \qquad n = 1, 2, \dots ,$$

and hence

$$\tau = \lim_{n\to\infty} \frac{x_n}{n};$$

(ii) if the number τ is rational of the form $\tau = p/q$ with coprime p and q then the map $f(\cdot)$ has a periodic point of period q, and any trajectory (29) converges to a periodic trajectory of period q as $n \to \infty$;

(iii) if the number τ is irrational then all the trajectories (29) have the same limiting set which is either coincide with the whole circle or is the Cantor set;

(iv) the number τ depends continuously on the graph of the map F in the Hausdorff metric[11].

According to this Theorem the number τ is uniquely determined by the map F and does not depend neither on the choice of the initial point x_0 of the trajectory $\{x_n\}$ nor on arbitrariness in the construction of the trajectory $\{x_n\}$ by formula (29). So, it is reasonable to denote the number τ by $\tau(F)$; this number is called *the rotation number* of the lift F. The value $\tau(F)$ is often called also the rotation number of the circle map f. One should only bear in mind that the rotation number for a circle map is defined modulo integer additives since lifts of the circle map are also defined modulo integer additives. Therefore, sometimes the rotation number of a circle map is defined as $\tau(F) \pmod 1$.

Remark 2. An orientation preserving circle map was defined above as such a circle map which has a strictly increasing lift. Theorem 7 will be no longer valid if to omit the requirement that the corresponding lift increases strictly.

Proof. Validity of the remark follows from the fact that a circle map with a non-decreasing lift may have simultaneously periodic points of different coprime periods as is plotted in Fig. 2 and 3. □

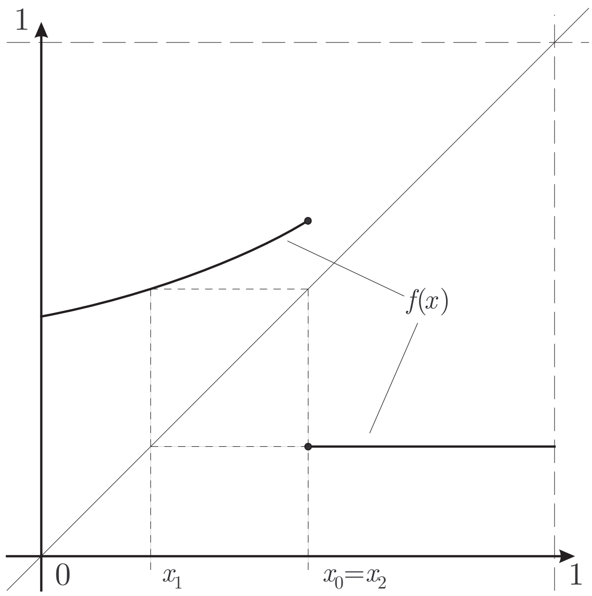

Figure 2. Periodic point of period 2

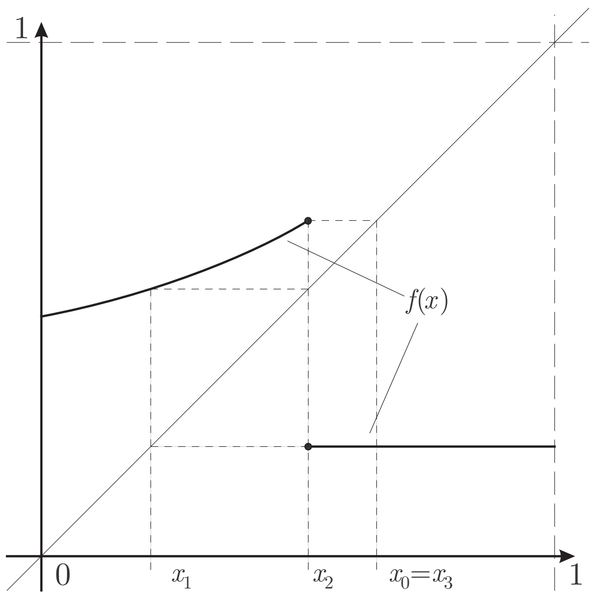

Figure 3. Periodic point of period 3

The next Remark shows that in Theorem 7 the requirement of the connectedness of the graph of the lift F is not essential. What is important is the closeness of the graph.

Remark 3. All the statements of Theorem 7 continue to be valid for any circle map possessing a strictly increasing closed lift.

Proof. Let the circle map $f(x)$ has a strictly increasing closed lift $F(x)$. Consider the connected closure $F_c(x)$ of the map $F(x)$. Then from the inclusions $F(x) \subseteq F_c(x)$ valid for any $x \in \mathbb{R}$ it

[11] The statement means that for any orientation-preserving circle map $\hat{f}$ with a connectedly closed lift $\hat{F}$ the values of $\hat{\tau}$ tend to τ when the graph of the map $\hat{F}$ tends to the graph of the map F by the Hausdorff metric. Point out that due to condition (27) the Hausdorff distance between the maps F and $\hat{F}$ is defined correctly in spite of the fact that the graphs of these maps are not bounded.

follows that each trajectory $\{x_n\}$ of the map $F(x)$ is also a trajectory of the map $F_c(x)$. Hence, the rotation number $\tau(F)$ of the map F is correctly defined and coincides with $\tau(F_c)$, and besides, the limiting set of the trajectory $\{x_n\}$ does not depend on the choice of the trajectory in the case when $\tau(F)$ is irrational.

If the number $\tau(F)$ is rational then the trajectory $\{x_n\}$ of the map F, being at the same time a trajectory of the map F_c, by assertion (iii) of Theorem 7 converges to some periodic trajectory of the map F_c. But in view of closeness of the graph of the map F the corresponding limiting trajectory will be a trajectory of the map F, from which assertion (iii) of Theorem 7 for the map F follows.

At last, assertion (iv) of Theorem 7 for the map F follows from the already established identity $\tau(F) \equiv \tau(F_c)$ and from the remark that for any two strictly increasing maps F and $\hat{F}$ with the closed graphs the Hausdorff distance between their graphs coincide with the Hausdorff distance between the graphs of the maps F_c and $\hat{F}_c$. □

8. COMPUTATION OF THE ROTATION NUMBER

One of weak points in the definition of the rotation number $\tau(f)$ for the circle map $f(\cdot)$ is that one need perform intermediate steps (such as to construct the lift $F(\cdot)$ and to build the trajectory $\{x_n\}$ of the map $F(\cdot)$) to calculate the limit

$$\tau(f) = \lim_{n\to\infty} \frac{x_n}{n},$$

where $x_n \in F^n(x_0)$. It is desirable to find a method to calculate the rotation number $\tau(f)$ directly in terms of the map f and its trajectories. To do it, we first investigate in more details properties of the orientation-preserving circle maps (cf. [6, Lemma 1]).

Lemma 8. *Let f be a close orientation-preserving circle map and let F be its standard lift. Then for any $x \in [0,1)$ and any pair of elements $f_x \in f(x)$, $F_x \in F(x)$ satisfying $f_x = F_x \pmod 1$ the following relation is valid:*

$$F_x = f_x + \nu(f_x), \tag{30}$$

where

$$\nu(x) = \begin{cases} 1 \text{ if } & 0 \le x < \omega, \\ 0 \text{ if } & \omega \le x < 1, \end{cases} \tag{31}$$

with $\omega = \min\{y : \ y \in f(0)\}$ (see Fig. 1)[12].

Conversely, if for a pair of elements $f_x \in f(x)$ and F_x relation (30) holds then $F_x \in F(x)$.

Proof. Fix a point $x \in [0,1)$ and choose a pair of elements $f_x \in f(x)$ and $F_x \in F(x)$ satisfying the relation $f_x = F_x \pmod 1$. Since, by the lemma's conditions, $F(\cdot)$ ia a standard lift of the map $f(\cdot)$ then $F(0) = f(0) \in [0,1)$. Then from the fact that the map $F(\cdot)$ is strictly increasing we obtain the estimates

$$0 \le f(0) = F(0) \le F_x < F(1) = F(0) + 1 = f(0) + 1 < 2, \qquad x \in [0,1),$$

i.e. $F_x \in [0,2)$.

If $F_x \in [0,1)$ then the equality $f_x = F_x \pmod 1$ implies $f_x = F_x$, and by monotony of the function $F(\cdot)$

$$\omega = \min\{y : \ y \in f(0)\} = \min\{y : \ y \in F(0)\} \le F_x = f_x < 1.$$

[12] Remark that the function $\nu(x)$ is identically equal to zero if $\omega = 0$. In this case $F(x) \equiv f(x)$ on the interval $[0,1)$, and so, the function $f(x)$ strictly increases on $[0,1)$.

Hence, in this case $\nu(f_x) = 0$ from which we obtain that $F_x = f_x + \nu(f_x)$.

But if $F_x \in [1, 2)$ then the equality $f_x = F_x \pmod 1$ implies $f_x = F_x - 1$. In this case by monotony of the function $F(\cdot)$ the following relations take place

$$0 \le f_x = F_x - 1 < \min\{y : \ y \in F(1)\} - 1 = \min\{y : \ y \in F(0) + 1\} - 1 =$$
$$= \min\{y : \ y \in F(0)\} = \min\{y : \ y \in f(0)\} = \omega.$$

Hence $\nu(f_x) = 1$ which again implies $F_x = f_x + \nu(f_x)$. So, in one direction Lemma 8 is proved.

Now, let $f_x \in f(x)$ and F_x be elements for which relation (30) is fulfilled. By the definition of the lift of a circle map, the sets $f(x)$ and $F(x)$ satisfy the relation $f(x) = F(x) \pmod 1$. Consequently, the set $F(x)$ contains such an element F_* that $f_x = F_* \pmod 1$. But then, due to the already proven first part of Lemma, the relation $F_* = f_x + \nu(f_x)$ should be valid. But by supposition, for the elements f_x and F_x the analogous relation (30) is also true, i.e. $F_x = f_x + \nu(f_x)$, from which we immediately obtain $F_x = F_* \in F(x)$. Lemma 8 is completely proved. □

At last, we are able to present the definition of the rotation number of the circle map $f(\cdot)$ directly in terms of the map $f(\cdot)$ (to be precise, the definition of the rotation number of the standard lift $F(\cdot)$ of the map $f(\cdot)$).

Theorem 8. *Let $f : [0, 1) \to [0, 1)$ be an orientation-preserving circle map with the closed standard lift F. Let $\{z_n\}$ be a trajectory of the map f, i.e.*

$$z_{n+1} \in f(z_n), \qquad n = 0, 1, \dots \ .$$

Then the uniform estimates hold

$$\left| \frac{\sum_{i=1}^{n} \nu(z_i)}{n} - \tau(F) \right| \le \frac{2}{n}, \qquad n = 1, 2, \dots \ , \tag{32}$$

and so,

$$\tau(F) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \nu(z_i)}{n}.$$

Proof. Define the sequence $\{x_n\}_{n=0}^{\infty}$ by setting $x_0 = z_0$ and

$$x_n = z_n + \sum_{i=1}^{n} \nu(z_i), \qquad n = 1, 2, \dots \ .$$

Prove by induction that $\{x_n\}$ satisfies the inclusions

$$x_{n+1} \in F(x_n), \qquad n = 0, 1, \dots \ , \tag{33}$$

and so, it is a trajectory of the map F.

Indeed, by the definition, $x_1 = z_1 + \nu(z_1)$, where $z_1 \in f(z_0)$. Therefore, by Lemma 8 $x_1 \in F(z_0) = F(x_0)$, and the statement of Theorem 8 is true for $n = 0$.

Perform the step of induction. Suppose that the statement of Theorem 8 is valid for $n = k \ge 0$ and show that this imply its validity for $n = k + 1$. By the definition of the element x_{k+1},

$$x_{k+1} = z_{k+1} + \sum_{i=1}^{k+1} \nu(z_i)$$

or, what is the same,

$$x_{k+1} - \sum_{i=1}^{k} \nu(z_i) = z_{k+1} + \nu(z_{k+1}).$$

Since here, by the definition of the trajectory $\{z_n\}$, the inclusion $z_{k+1} \in f(z_k)$ with $z_k \in [0,1)$ holds, then by Lemma 8 $z_{k+1} + \nu(z_{k+1}) \in F(z_k)$. Hence,

$$x_{k+1} - \sum_{i=1}^{k} \nu(z_i) \in F(z_k)$$

or, what is the same,

$$x_{k+1} \in F(z_k) + \sum_{i=1}^{k} \nu(z_i) = F(z_k + \sum_{i=1}^{k} \nu(z_i)).$$

Here, by the supposition of induction, the argument of the function F in the right-hand part coincides with x_k which implies $x_{k+1} \in F(x_k)$.

So, the step of induction is justified and inclusions (33) are proved. To complete the proof of Theorem 8 it remains to note only that by Theorem 7 and Remark 3 for the trajectory $\{x_n\}$ the estimates hold

$$\left| \frac{x_n}{n} - \tau(F) \right| \le \frac{1}{n}, \qquad n = 1, 2, \dots\ ,$$

while by the definition of trajectory $\{x_n\}$ it is valid the equality

$$\frac{x_n}{n} = \frac{z_n}{n} + \frac{\sum_{i=1}^{n} \nu(z_i)}{n},$$

where $z_n \in [0,1)$. Estimates (32) now directly follow from the latter relations. Theorem 8 is proved. □

REFERENCES

1. E. A. Asarin, V. S. Kozyakin, M. A. Krasnosel'skii, and N. A. Kuznetsov, *Analysis of the stability of asynchronous discrete systems.* Moscow: Nauka, 1992, in Russian. 283

2. T. Bousch and J. Mairesse, "Asymptotic height optimization for topical IFS, Tetris heaps, and the finiteness conjecture," *J. Amer. Math. Soc.*, vol. 15, no. 1, pp. 77–111, 2002. 283

3. R. Brette, "Rotation numbers of discontinuous orientation-preserving circle maps," *Set-Valued Analysis*, vol. 11, no. 4, pp. 359–371, 2003. 284, 295, 296

4. A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dymanical Systems*, ser. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1992. 283, 284, 285, 286, 288, 289, 291

5. V. Kozyakin, "Proof of a counterexample to the finiteness conjecture in the spirit of the theory of dynamical systems," Weierstraß-Institut für Angewandte Analysis und Stochastik, Berlin, Preprint 1005, Jan. 2005. 283

6. V. S. Kozyakin, "Sturmian sequences generated by order preserving circle maps," Boole Centre for Research in Informatics, University College Cork — National University of Ireland, Cork, Preprint 11/2003, May 2003. 284, 285, 298

7. Z. Nitecki, *Differentiable dynamics.* Cambridge, Massachusetts and London, England: The MIT Press, 1971. 283, 284, 286