======= **АНАЛИЗ И СИНТЕЗ СИСТЕМ УПРАВЛЕНИЯ** =======

Random Matrices, Information Theory and Physics: New Results, New Connections

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Abstract—Random matrix theory has found many applications in physics, statistics and engineering since its inception. In this paper we find new results and new connections between random matrices, information theory and physics. The contributions of this paper are:
1) Recursive formula to evaluate the expectation of the first moment (correlation function) of the characteristic polynomial of non-Hermitian matrix
2) It is shown that expectation of an expression of the form $E_H \det(I + \alpha H)$ with Gaussian Unitary Ensemble (GUE) satisfies KP (Kadomtsev-Petviashvili) equations. Same is true for Wishart matrices
3) Connection between expected value of the expression of form, $E_W \det(I + \alpha W)$, (where W is Wishart matrix) and Toda lattice equation. Besides, we discuss some beautiful connections established in [31,32,33] between random matrices and Painleve differential equations and Toda lattice equations.

1. INTRODUCTION

Random Matrix Theory (RMT) has found many applications in physics, statistics and engineering. Physicists usually think of Wigner and Dyson as fathers of RMT [13]. However, twenty years before their work on the subject, Wishart [19] examined random matrices of the form HH^H (H is a matrix with Gaussian entries) as a tool for studying multivariate data. The properties of these so-called Wishart matrices, which are viewed as "fundamental to multivariate statistical analysis also find important applications in fields from information theory and communications, to mesoscopics [20], to high energy physics [21], to econo-physics [22] etc. Although early developments were motivated by practical experimental problems, random matrices are now used in fields as diverse as Riemann hypothesis, stochastic differential equations, condensed matter physics, statistical physics, chaotic systems, numerical linear algebra, etc. Graph theory, classical compact groups, orthogonal polynomials, integral equations, non-commutative probability theory, combinatorics are enriched due to the studies of the random matrix properties. Functions with zeroes displaying the statistics of the eigenvalues of random matrices occur in number theory, for example the Riemann zeta function and L-functions [14, 26].

Recent research in wireless communications has focused on potential performance gains achievable by using multiple element arrays at both the transmitter and receiver. For example, the use of multiple antennas is currently under consideration in the 802.11n standardization efforts. The reason for this interest is that in theory, much larger spectral efficiency can be achieved by utilizing spatial diversity at both the transmitter and the receiver as compared to the single-input-single-output (SISO) case. In this paper we give new results and new connections between RMT, MIMO information theory, and Physics. The paper is organized as follows. In Section 2, we give signal model and gives ergodic capacity of MIMO system. Section 3 is devoted to Grassmann variables and its properties. In section 4 we give replica treatment to evaluate capacity of uncorrelated MIMO system. Relationship between expression of form $E_W[\det(\alpha + W)^N]$, (where W is Wishart matrix and E_W is expectation operator) and Toda lattice equation and also to Painleve differential

equation is given in section 5 and section 6 respectively [32,33]. In Section 7 we show the method to evaluate recursively the expected value of (correlation function) characteristic polynomial of non-Hermitian matrix. Section 8 gives the connection between expected value of the form $E[\det(I + \alpha W)]$ to a different form of Toda lattice equation. In section 9 it is shown that the expression of the form $E[\det(I + \alpha H)]$ with Gaussian Unitary Ensemble (GUE) satisfies KP (Kadomtsev-Petviashvili) equations. Same is true for Wishart matrices. The KP equations [23, 24] occur maturally in many physical contexts as "universal" models for the propagation of weakly nonlinear dispersive long waves which are essentially one-directional, with weak transverse effects. The KP equations are two dimensional extensions of the Korteweg-de Vries (KdV) equation, which is the first known soliton equation. We give conclusions in the last section.

2. SYSTEM MODEL

We will consider a discrete-time flat-fading channel model with N receive and N transmit antennas. The motivation for such a model is a point-to-point wireless communication link. We assume that communication occurs on a burst basis, and that the channel can be modeled as linear and time-invariant during a burst. Hence, a simplified discrete-time baseband model of the system can be written as:

$$y_j = \sum_{i=1}^{N} h_{ij} x_i + v_j, \tag{1}$$

where y_j is the received signal at receiver antenna j, reflecting contributions from all transmit antenna signals x_i, and v_j represents the additive white Gaussian noise seen at each receiver antenna. We will assume (without loss of generality) that the noise has unit generalized variance (i.e., $E[vv^T] = I_N$). The signal vector received at the output can be written in matrix form as

$$\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} h_{11} & \ldots & h_{N1} \\ \vdots & \vdots & \vdots \\ h_{1N} & \ldots & h_{NN} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix},$$

or in vector notation we have,

$$y = Hx + v. \tag{2}$$

It is assumed that the channel is stationary, ergodic and independent of the channel inputs x and the noise v. The following notations are used in the paper: I_N represents the $N \times N$ identity matrix, $Tr(X)$ is the trace of X, $\det Y$ is the determinant of Y, and $\bigotimes$ denotes the Kronecker product.

We assume that the overall input power at the transmitter is constrained to ρ,

$$Tr(E[xx^H]) \leq \rho. \tag{3}$$

We shall now use this linear model to derive the capacity of a MIMO communication system, under certain further assumptions on the system parameters.

2.1. Ergodic capacity

We first assume that H is a Gaussian random matrix whose realization is known at the receiver, or equivalently, that the channel output consists of the pair y, H. The input power is distributed equally over all transmitting antennas. Assuming a block fading model, then it is known that the ergodic capacity of a random MIMO channel is given by [1]:

$$C = E_H\{\log \det(I_N + \frac{\rho}{N} H^H H)\}, \tag{4}$$

where E_H denotes that the expectation is taken with respect to the ensemble statistics of H, which are Gaussian distributed in our case, i.e, $H \sim N(0, I_N \bigotimes I_N)$, and hence the envelopes are Rayleigh distributed. This is not an unreasonable model for a mobile communications environment.

3. INTRODUCTION TO GRASSMANN VARIABLES AND ALGEBRA

One mathematical technique which can assist in finding an approximate expression for the RHS of Eq. 4 is the use of Grassmann variables. We have already used Grassmann variables in evaluating capacity of MIMO systems in [25] but for the sake of completeness we write about it in this paper too. In this section, we will first provide an extremely abbreviated introduction to the concept and use of Grassmann variables. A more complete overview of Grassman algebra can be found in [2].

At their simplest, Grassmann variables are mathematical objects which obey the following anti-commutation rule:

$$\{\theta_i, \theta_j\} = \theta_i\theta_j + \theta_j\theta_i = 0, \quad for \ all \ i, j. \tag{5}$$

The anti-commuting rule of Eq.5 holds in particular for $i = j$, and hence we see that the square of an arbitrary variable θ_i is zero in this algebra,

$$\theta_i^2 = 0. \tag{6}$$

Note that these defining properties of Grassmann variables are mathematical conventions, and we do not attempt to ascribe any physical interpretation to such variables. Because of the property expressed in Eq.6, any function f of the anti-commuting variables is a finite polynomial. A further convention in Grassmann algebra is that the complex conjugate of a product of variables $\theta_1, \theta_2, \ldots, \theta_n$, is taken to be the product of the complex conjugates of these variables.

$$(\theta_1\theta_2\ldots\theta_n)^* = \theta_1^*\theta_2^*\ldots\theta_n^*. \tag{7}$$

We shall also need to define the complex conjugate of a complex conjugate $(\theta_i^*)^*$. In principle, one has different options and it is matter of taste which one is chosen. In this paper, we shall define it thus:

$$(\theta_i^*)^* = -\theta_i. \tag{8}$$

At first glance, the definition in Eq.8 is counter-intuitive, since it differs in sign on the right-hand side from the corresponding definition in conventional algebra. However, it is quite convenient for anti-commuting variables. For example, for the quantity $\theta_i^*\theta_i$, we have

$$(\theta_i^*\theta_i)^* = -\theta_i\theta_i^* = \theta_i^*\theta_i. \tag{9}$$

We see from this equation that the quantity $\theta^*\theta$ does not change under complex conjugation and therefore can be considered as "real", which is consistent with our expectations.

It is also straightforward to define linear combinations of the set of anti-commuting variables $\gamma_i, i = 1, \ldots, n$, to produce new variables θ_i:

$$\theta_i = \sum_{k=1}^{n} a_{ik}\gamma_k, \tag{10}$$

where the a_{ik} represent conventional (non-Grassmann) algebraic scalars. By calculating the product of different θ_i we can verify the following identity:

$$\theta_1\theta_2\ldots\theta_n = \gamma_1\gamma_2\ldots\gamma_n \det A, \tag{11}$$

where $A_{i,k} = a_{ik}$. The definition of derivatives of Grassmann variables [2] also proceeds in an expected fashion. However, the definition of integrals over Grassmann variables requires a little care, as explained further in the next subsection.

3.1. Integrals over Grassmann variables

Let us consider integrals over Grassmann variables, as first introduced by Berezin [5]. They are defined formally as follows:

$$\int d\theta_i = \int d\theta^* = 0, \tag{12}$$

$$\int \theta_i d\theta_i = \int \theta_i^* d\theta_i^* = 1. \tag{13}$$

The definition in Eq. 12, 13 is completely formal. The notation $\int$ is only a symbol, and one should not try to imagine this integral as representing an infinite sum. This definition implies that the "differentials" $d\theta_i, d\theta_i^*$ anti-commute with each other and with the variables θ_i, θ_i^*.

$$\{d\theta_i, d\theta_i\} = \{d\theta_i, d\theta_i^*\} = \{d\theta_i^*, d\theta_i^*\} = 0, \tag{14}$$

$$\{d\theta_i, \theta_i\} = \{d\theta_i, \theta_i^*\} = \{d\theta_i^*, \theta_i^*\} = 0. \tag{15}$$

The definition in Eq. 12, 13 is sufficient to introduce integrals of an arbitrary function. If a function depends on one variable θ_i it must be linear in θ_i because already $\theta_i^2 = 0$ (and higher order powers). Assuming that the integral of the sum of two functions equals the sum of the integrals we can calculate the integral of the function with Eq. 13. Repeated integrals are considered to be integrals over several variables. This enables us to calculate the integral of a function of an arbitrary number of variables. Note that the choice of unity on the right-hand side of Eq. 13 is completely arbitrary; one could write any finite number.

An important result for our later calculations is the formula for the integral of the Gaussian exponential of multiple Grassmann variables [2] given by,

$$\int \exp(-\theta^* A\theta) \prod_{i=1}^{n} d\theta_i^* d\theta_i = \det A, \tag{16}$$

where A is an $n \times n$ matrix whose entries are conventional (non-Grassmann) algebraic scalars and θ is the column vector composed of Grassmann variables. We also note that

$$\int \exp(-\theta_i^* b\theta_i) d\theta_i^* d\theta_i = b. \tag{17}$$

4. APPLICATION OF REPLICA ANALYSIS AND GRASSMANN VARIABLES TO CAPACITY EVALUATION

Before returning to the subject of MIMO capacity, we will also briefly introduce another technique found in theoretical physics termed replica analysis [3, 4]. In 1975, Edwards and Anderson, when studying disordered systems of spins, proposed a new method for the investigation of disordered systems – the so called replica method. In this method, one replaces a single disordered system by n systems which are identical to the original. Then, for example, instead of calculating the free energy, $F = T \log Z$, one calculates the quantity, $F_n = T(\frac{d}{dn} Z^n)$. Where $\frac{d}{dn}$ is derivative with respect to n. The limit of F_n as $n \to 0$ coincides with the free energy F. Mathematically this is represented by,

$$F = T \log Z. \tag{18}$$

The above expression can be calculated as

$$F = T \left(\lim_{n \to 0} \frac{d}{dn} Z^n \right) = T \left(\lim_{n \to 0} \frac{(Z^n - 1)}{n} \right). \tag{19}$$

Recall, that the capacity of uncorrelated MIMO case is given by

$$C = E_H \{ \log \det(I_N + \frac{\rho}{N} H^H H) \}.$$

At relatively high SNR we have

$$C = E_H \{ \log \det(\frac{\rho}{N} H^H H) \}. \tag{20}$$

Firstly, borrowing from replica analysis, we can rewrite the capacity expression (at relatively high SNR)as

$$C = \lim_{n \to 0} \frac{d}{dn} E_H \{ \det(\frac{\rho}{N} H^H H)^n \}. \tag{21}$$

We can write

$$\det(\frac{\rho}{N} H^H H) = \det(\sqrt{\frac{\rho}{N}} H) \det(\sqrt{\frac{\rho}{N}} H^H) \tag{22}$$

We will now use Grassmann algebra to evaluate the determinant term. Introducing two sets of Grassmann variables $\psi_{ai}, \bar{\psi}_{ai}, \chi_{ai}, \bar{\chi}_{ai}$. Using Eqs.16,21,22, we have

$$G = E_H \{ \det \frac{\rho}{N} H^H H)^n \} = \int d\mu(H)$$

$$\int d\mu(\psi, \chi) \exp(-\sqrt{\frac{\rho}{N}} \bar{\chi}_a H^H \chi_a - \sqrt{\frac{\rho}{N}} \bar{\psi}_a H \psi_a \}) \tag{23}$$

where

$$d\mu(H) = (\pi)^{-N^2} \exp(-Tr(H^H H)) \prod_{m=1}^{2} \prod_{i,j=1}^{N} dH_{ij}^{(m)} \tag{24}$$

where $d\mu(\psi, \chi) = \prod_{i,a=1}^{N} d\psi_{ai} d\chi_{ai}$, is the integration measure, the ψ_a is a vector of Grassmann variables and $\bar{\psi}_a$ is the (Hermitian) complex conjugate of ψ_a.

By carrying out integration with respect to H, we arrive at

$$G = \int d\mu(\psi, \chi) exp\{ \frac{\rho}{N} \bar{\psi}_{ai} \psi_{aj} \bar{\chi}_{bj} \chi_{bi} \} \tag{25}$$

Note that by carrying out integration with respect to H we generate quartic terms in Grassmann variables. We introduce an auxiliary matrix Q_{ab} to decouple the quartic term in Grassmann variables (a standard technique used in statistical physics) as

$$G = lim_{\epsilon \to 0} \int_{C^{n \times n}} dQ exp(-Tr(QQ^H)) \int d\bar{\psi} d\psi d\bar{\chi} d\chi$$

$$\exp -\{ (\bar{\psi}_{ai} \ \bar{\chi}_{ai}) \begin{pmatrix} \epsilon \delta_{ab} & -\sqrt{\frac{\rho}{N}} Q_{ab} \\ \sqrt{\frac{\rho}{N}} Q_{ab}^H & \epsilon \delta_{ab} \end{pmatrix} \begin{pmatrix} \psi_{bi} \\ \chi_{bi} \end{pmatrix} \}$$

where δ_{ab} is the Kronecker delta. Carrying out trivial Grassmann integration [4], we obtain

$$G = lim_{\epsilon \to 0} \int_{C^{n \times n}} dQ exp(-Tr(QQ^H)) \det \begin{pmatrix} \epsilon I & -\sqrt{\frac{\rho}{N}} Q \\ \sqrt{\frac{\rho}{N}} Q^H & \epsilon I \end{pmatrix}^N$$

which can be further written as (after simplification and ignoring a term which approaches unity in the replica limit) and also taking limit of ϵ,

$$G = \int_{C^{n \times n}} dQ exp(-TrQQ^H) \det(\frac{\rho}{N}QQ^H)^N \tag{26}$$

where we have used the following property of determinants:

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC) \quad \text{if} \quad CD = DC \tag{27}$$

which can be further written as (after simplification and ignoring a term which approaches unity in the replica limit)

$$G = \int_{C^{n \times n}} dQ exp(-TrQQ^H) \det(\frac{\rho}{N}QQ^H)^N, \tag{28}$$

where A, B, C, D are block matrices of appropriate dimensions, and where $dQ = \pi^{-n^2} \prod_{a,b}^n d^2 Q_{ab}$ and $d\psi = \prod_{a,i} d\psi_{ai}$, with a similar measure defined for χ. A complex matrix $Q \in C^{n \times n}$ can be uniquely written using the singular value decomposition as $Q = U\Sigma V^H$, where $U \in U(n)$ and $V \in V(n)$ are unitary matrices and λ_a are the eigenvalues of QQ^H. The Euclidean measure dQ on $C^{n \times n}$ is related to the normalized Haar measures dU on $U(n)$ and dV on $V(n)$ by

$$dQ = dUdV\Delta(\lambda)^2 \prod_{a=1}^n d\lambda_a, \tag{29}$$

where $\Delta\lambda = \prod_{a>b}^n (\lambda_a - \lambda_b)$ is Vandermonde determinant. Carrying out the integration over $\mathbf{U}$ and V and using the property of Haar measures, i.e., $\int dU = \int dV = 1$, we obtain

$$G = \int_0^\infty \prod_{a=1}^n d\lambda_a \exp(-\lambda_a)(\frac{\rho}{N}\lambda_a)^N \Delta(\lambda)^2 \tag{30}$$

up to an irrelevant constant factor that approaches unity in the replica limit. The above equation can further be written as

$$G = \int_0^\infty \prod_{a=1}^n d\lambda_a \exp\{-\lambda_a + N\ln(\frac{\rho}{N}\lambda_a)\}\Delta(\lambda)^2. \tag{31}$$

Upto this point we are in parallel with [25]. The large "N" limit of the above integral was performed in [25] to give simple expression of the uncorrelated MIMO system capacity.

5. RELATIONSHIP BETWEEN $E[\det(\alpha + W)^N]$ AND TODA EQUATIONS

The material in this section and section.6 had already appeared in [32,33]. Consider the following equation (in this section, $(n, N) \geq 1$)

$$Z_w = E_W \det(\alpha + W)^N \tag{32}$$

where $W = H^H H$ is Wishart matrix and H is $n \times n$ (square) matrix. Performing eigen decomposition, the above equation can be written as (upto a constant factor)

$$Z_w = \int_0^\infty \prod_{a=1}^n d\lambda_a \exp(-\lambda_a)(\alpha + \lambda_a)^N \Delta(\lambda)^2 \tag{33}$$

Now we show that Toda Lattice Hierarchy [7, 27, 28, 29] emerges from above equation. To proceed we represent Vandermonde determinant as [16, 12, 15]

$$\prod_{a_1 < a_2 = 1}^{n} (\lambda_{a_1} - \lambda_{a_2}) = \det(\lambda_k^{l-1})_{k,l=1,\dots n}, \tag{34}$$

simultaneously shifting all $\lambda_a' s$ therein by α, and perform the n-fold integral by means of the Binet-Cauchy formula [6], i.e.,

$$\int \prod_{l=1}^{n} d\mu(\lambda_l) \det[(A_k(\lambda_l)]_{k,l=1,\dots,n} \det[B_k(\lambda_l)]_{k,l=1,\dots,n}$$

$$= n! \det \left(\int d\mu(\lambda) A_k(\lambda) B_l(\lambda) \right)_{k,l=1,\dots n} \tag{35}$$

using the above formula, the integral is given by (upto constant factor)

$$Z_w = \det \left(\int_0^\infty d\lambda e^{-\lambda} (\lambda + \alpha)^{N+k+l} \right)_{k,l=0,\dots,n-1} \tag{36}$$

The above equation can be further written as

$$Z_w = e^{n\alpha} (\alpha)^{n(n+N)} \tilde{\tau}_n(\alpha; N) \tag{37}$$

which involves Hankel determinant [15, chapter.3].

$$\tilde{\tau}_n(\alpha; N) = \det[\partial_{(\alpha)}^{k+l} \tilde{\tau}_1(\alpha; N)]_{k,l=0,\dots,n-1} \tag{38}$$

with

$$\tilde{\tau}_0(\alpha; N) = 1$$

$$\tilde{\tau}_1(\alpha; N) = \int_1^\infty d\lambda \lambda^N e^{-\alpha\lambda} = \frac{\Gamma(N+1, \alpha)}{\alpha^{N+1}} \tag{39}$$

Here, $\Gamma(a, x)$ is upper incomplete gamma function and is given by

$$\Gamma(a, x) = \int_x^\infty dt\, t^{a-1} e^{-t} \tag{40}$$

The Hankel determinant Eq. 38 is remarkable object. Whatever the function $\tilde{\tau}_1(\alpha; N)$ is, by virtue of the Darboux Theorem, the entire sequence $\{\tilde{\tau}_{k \in Z^+}\}$ satisfies the equation (see for example)[30]

$$\tilde{\tau}_n \tilde{\tau}_n'' - (\tilde{\tau}_n')^2 = \tilde{\tau}_{n-1} \tilde{\tau}_{n+1}, \quad n \in Z^+ \tag{41}$$

where prime $'$ stands for ∂_α. Eq. 41 is known as Toda Lattice equation in the theory of integrable hierarchies [7, 27, 28]. Hence establishing a link between $E[\det(\alpha + W)^N]$ and Toda lattice equations.

6. RELATIONSHIP BETWEEN $E[\det(\alpha + W)^N]$ AND PAINLEVE DIFFERENTIAL EQUATIONS

This connection was established in [33] but we discuss it here for the sake of completeness of the paper. The connection between probability distribution function of the eigenvalues of random matrices and Painleve differential equations was shown by Tracy and Widom (see for example [12]). Considering Eq. 33, observing

that the n-fold integral is essentially a Fredholm determinant [8, 9, 10, 31] associated with a gap formation probability

$$E_n^{(0,\alpha)}(0;a) = \int_\alpha^\infty \prod_{l=1}^n d\lambda_l e^{-\lambda_l} \lambda_l^a \prod_{l_1<l_2=1}^n (\lambda_{l_1} - \lambda_{l_2})^2 \tag{42}$$

within the interval $(0,\alpha)$ in the spectrum of auxiliary $n \times n$ Laguerre unitary ensemble. In [11] one of the result states that

$$E_n^{(0,\alpha)}(0;a) = exp\Big(\int_0^\alpha dt \frac{\sigma_5(t)}{t} \Big) \tag{43}$$

where $\sigma_5 = \sigma_n(t;a)$ is the fifth Painleve transcedent satisfying the Jimbo–Miwa–Okamoto form of the Painleve 5 equation [10]

$$(t\sigma_5'')^2 - (a\sigma_5')^2 - (\sigma_5 - t\sigma_5')[\sigma_5 - t\sigma_5' + 4\sigma_5'(\sigma_5' + n + \frac{a}{2})] = 0 \tag{44}$$

with boundary condition [12]

$$\sigma_5(t)|_{t\to\infty} \sim -nt + an - \frac{an^2}{t} + O(t^{-2}). \tag{45}$$

In Eq. 33 shifting all eigenvalues by α, an operation which leaves the Vandermonde determinant unchanged. We arrive at Painleve representation of our integral

$$Z_w = e^{n\alpha} exp\Big(\int_0^\alpha dt \frac{\sigma_5(t)}{t} \Big) \tag{46}$$

where $\sigma_5(t) = \sigma_n(t;a=N)$. Hence we find an alternative representation of $E[\det(\alpha + W)^N]$ in terms of fifth painleve differential equation [32,33].

7. RECURSIVE FORMULA TO EVALUATE THE FIRST MOMENT OF CHARACTERISTIC POLYNOMIAL OF GAUSSIAN UNITARY ENSEMBLE

The recursive form can be obtained by considering the following equation (in this section H is $N \times N$ matrix).

$$Z_N = \lim_{n\to 1} \int_{C^{N\times N}} dH e^{-Tr(H^H H)} \det(\alpha - H)^n \det(\bar\alpha - H^H)^n \tag{47}$$

where $dH = \pi^{-N^2} \prod_{i,j=1}^N d^2 H_{ij}$. By employing the same machinary of section.4 (see also [3]) we get (upto a constant factor)

$$Z_N = lim_{n\to 1} \int_{C^{n\times n}} dQ exp(-TrQ^H Q) \det(|\alpha|^2 + Q^H Q) \tag{48}$$

By performing eigen decomposition (as in section.4) and putting the limit, using the fact that Vandermonde determinant is unity in the limit, we get

$$Z_N = C_N \int_0^\infty d\lambda \exp(-\lambda)(|\alpha|^2 + \lambda)^N. \tag{49}$$

where C_N is appropriate constant. Carrying out integration by parts we arrive at the required recursive formula.

$$\frac{C_N}{(N+1)(C_{N+1})} Z_{N+1} = Z_N + \frac{C_N |\alpha|^{2N+2}}{N+1} \tag{50}$$

8. CONNECTION BETWEEN $E[\det(I + \alpha W)]$ AND TODA LATTICE EQUATION

Consider the Wishart matrix as in previous section. Consider the following equation

$$Z_N = E_W \det(I + \alpha W) \tag{51}$$

where expectation is with respect to Wishart density ($W = H^H H$) and H is $N \times N$ matrix. This equation can also be written as

$$Z_N = \int dW e^{-Tr(W)} exp[Tr \ln(1 + \alpha W)] \tag{52}$$

where for the sake of simplicity we do not write the constant prefactor. In writing the above equation we have used the fact that

$$\det A = exp[Tr \ln A]$$

Z_N can further be written as (by employing power series expansion of logarithm)

$$Z_N = \int dW exp[Tr \sum_{k=1}^{\infty} t_k W^k], \tag{53}$$

where $t'_k s$ are appropriate coefficients of W^k. By employing the eigenvalue decomposition of the matrix W, we get

$$Z_N = C_N \int_0^{\infty} \prod_{l=1}^{N} d\lambda_l exp[\sum_{k=1}^{\infty} t_k \lambda_i^k] \prod_{i<j}^{N} (\lambda_i - \lambda_j)^2, \tag{54}$$

C_N is appropriate constant, $\prod_{i<j}^{N}(\lambda_i - \lambda_j)^2$ is the Jacobian of transformation (which is square of Vandermonde determinant). The above equation has exactly the same form as partition function in the theory of integrable systems [12, chapter 4)] except that integration domain in our case is $[0, \infty]$ as opposed to $[-\infty, \infty]$ in [12] and we have to choose orthogonal polynomial which is orthogonal on domain $[0, \infty]$ with respect to weight function $exp\{\sum_{k=1}^{\infty} t_k \lambda^k\}$. By carrying out similar analysis as in [12 chapter 4] we can show that $Z_N = E[\det(I + \alpha W)]$ satisfies Toda Lattice equation (another form of Toda Lattice equation), given by [12],

$$\frac{\partial^2}{\partial t_1^2} \ln Z_N = \frac{Z_{N+1} Z_{N-1}}{Z_N^2} \tag{55}$$

9. RELATIONSHIP BETWEEN $E[\det(I + \alpha H)]$ OF RANDOM HERMITIAN MATRIX WITH GUE AND KP EQUATIONS

Let H be Hermitian matrix of size $N \times N$, the eigenvalue pdf is given by [12,13]

$$\frac{1}{C_{\beta N}} exp(-\frac{\beta}{2} \sum_{j=1}^{N} \lambda_j^2) \prod_{j,k=1}^{N} (\lambda_k - \lambda_j)^{\beta} \tag{56}$$

with $\beta = 1, 2$ and 4 for Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE) and Gaussian Symplectic Ensemble (GSE) respectively. $C_{\beta N}$ is appropriate constant. Consider the following equation (for Hermitian GUE)

$$Z_{N1} = E\{\det(I + \alpha H)\} \tag{57}$$

It can also be written as

$$Z_{N1} = \int dH e^{-Tr(H^H H)} exp[Tr \ln(I + \alpha H)] \tag{58}$$

where for the sake of simplicity we do not write the constant prefactor. In writing the above equation we have used the fact that

$$\det A = exp[Tr \ln A]$$

Z_{N1} can further be written as (writing logarithm in its power series)

$$Z_{N1} = \int dH exp[Tr \sum_{k=1}^{\infty} t_k H^k], \tag{59}$$

where $t'_k s$ are appropriate coefficients of H^k. By employing the eigenvalue decomposition of the matrix H, we get

$$Z_{N1} = C(N) \prod_{l=1}^{N} d\lambda_l exp[\sum_{k=1}^{\infty} t_k \lambda_i^k] \prod_{i<j}^{N} (\lambda_i - \lambda_j)^2, \tag{60}$$

where $\prod_{i<j}^{N}(\lambda_i - \lambda_j)^2$ is the Jacobian of transformation and is square of Vandermonde determinant and $C(N)$ normalization constant. The Vandermonde determinant can be written as [12, 16]

$$\Delta(\lambda) = \prod_{1 \leq i,j \leq N} (\lambda_i - \lambda_j)$$

$$= \det \begin{pmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_N \\ \vdots & \vdots & \vdots \\ \lambda_1^{N-1} & \dots \lambda_N^{N-1} \end{pmatrix}. \tag{61}$$

Vandermonde determinant can also be expressed in terms of orthogonal polynomial. We make use of the orthogonal polynomial $P_n(\lambda)$, (orthogonal with respect to the measure or weight) [15, 16]

$$\int_{-\infty}^{\infty} d\lambda e^{-V(\lambda)} P_n(\lambda) P_m(\lambda) = \delta_{nm}, \tag{62}$$

where δ_{nm} is Kronecker delta. The $P'_n s$ are orthogonal polynomials and are functions of single real variable λ. If $P_m(\lambda) = \lambda^m + \sum_{j=0}^{m-1} P_{m,j} \lambda^j$ are monic polynomials of degree m, for $m = 0, 1, \ldots, N-1$, then

$$\Delta(\lambda) = \det(\lambda_j^{i-1})_{1 \leq i,j \leq N} = \det(P_{i-1}(\lambda_j))_{1 \leq i,j \leq N} \tag{63}$$

This is easily proved by performing suitable linear combinations of columns, a procedure that leaves the determinant unchanged. The determinant are indexed with $i, j \in \{1, 2, \ldots, N\}$. Hence the Vandermonde determinant can be written as (in terms of orthogonal polynomial)

$$\Delta(\lambda) = \det \begin{pmatrix} P_0(\lambda_1) & \dots & P_0(\lambda_N) \\ P_1(\lambda_1) & \dots & P_1(\lambda_N) \\ \vdots & \vdots & \vdots \\ P_{N-1}(\lambda_1) & \dots P_{N-1}(\lambda_N) \end{pmatrix} \tag{64}$$

Using Eq. 63 we can write Eq. 60 as,

$$Z_{N1} = C(N) \int_{-\infty}^{\infty} exp(\sum_{i=1}^{N} \sum_{k=1}^{\infty} t_k \lambda_i^k) \Delta(\lambda) \det(P_{j-1}(\lambda_i))_{ij} \prod_{i=1}^{N} d\lambda_i \tag{65}$$

It can be written as

$$Z_{N1} = N!C(N)\det\Big(\int_{-\infty}^{\infty} exp(\sum_{k=1}^{\infty} t_k\lambda^k)P_{j-1}(\lambda)\lambda^{i-1}d\lambda\Big)_{ij} \tag{66}$$

Eq. 66 follows by applying Cauchy-Binet formula (Eq.35) to Eq. 65. Let us now consider the relation between general graphs and connected graphs. Le $p_r(t)$ be the number of arbitrary degree j graphs of order r and t_v the number of connected degree j graphs of order r, where $j \geq 1$ is fixed number. Then we have [17]

$$p_r(t) = \sum_{n_1+2n_2+3n_3+...=r} \frac{t_1^{n_1}.t_2^{n_2}.t_3^{n_3}\ldots}{n_1!.n_2.n_3\ldots} \tag{67}$$

where n_i is the number of connected components with i vertices in a given graph. The above formula is equivalent to a simple functional relation in terms of generating function [17]:

$$\sum_{r=0}^{\infty} p_r(t)k^r = exp(\sum_{\alpha=1}^{\infty} t_\alpha k^\alpha) \tag{68}$$

Eq. 68 is also definition of Schur polynomial [27, 28, 29]. Substituting Eq. 68 into Eq. 66, we get

$$Z_{N1} = N!C(N)\det\Big(\int_{-\infty}^{\infty}\sum_{r=0}^{\infty} p_r(t)\lambda^r P_{j-1}(\lambda)\lambda^{i-1}d\lambda\Big)_{ij} \tag{69}$$

$$= N!C(N)\det\Big(\int_{-\infty}^{\infty}\sum_{r=0}^{\infty} p_{r-i+1}(t)\lambda^r P_{j-1}(\lambda)d\lambda\Big)_{ij}$$

$$= N!C(N)\det\Big(\sum_{r=0}^{\infty} p_{r-i+1}(t)\int_{-\infty}^{\infty}\lambda^r P_{j-1}(\lambda)d\lambda\Big)_{ij}$$

$$= \det\Big(\sum_{r=0}^{\infty} p_{r-i+1}(t)\beta_{rj}\Big)_{ij} \tag{70}$$

where we have defined

$$\beta_{rj} = N!C(N)\int_{-\infty}^{\infty}\lambda^r P_{j-1}(\lambda)d\lambda. \tag{71}$$

It has been proven in [18] that determinant of the form

$$\det\Big(\sum_{r=0}^{\infty} p_{r-i+1}(t)\beta_{rj}\Big)_{ij} \tag{72}$$

satisfies the Hirota bilinear form of the KP equations, hence establishing the result. It is straight forward to show the same result for Wishart matrices.

10. CONCLUSIONS

In this paper we showed different connections between random matrices, MIMO information theory and Physics. In particular we showed that a recursive formula to evaluate the expectation of (correlation function) of the characteristic polynomial of non-Hermitian matrix. It is shown that expression of form, $E[\det(I + \alpha H)]$, of random Hermitian matrix with Gaussian Unitary Ensemble (GUE) satisfies KP (Kadomtsev-Petviashvili) equations. Same is true for Wishart matrices. We also established the connection between expression of form, $E[\det(I + \alpha W)]$ (where W is Wishart matrix) and Toda lattice equation.

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