Информационные процессы, Том 6, № 3, 2006, стр. 256–257.

On Asymptotic Property of the Sojourn Time in the M/G/1—EPS Queue [1]

S.F.Yashkov

Institute for Information Transmission Problems, 19, Bolshoi Karetny lane, 127994 Moscow GSP–4, Russia. E-mail: yashkov@iitp.ru

Received June 22, 2006

Abstract—Here is given the theorem concerning the light–tail behaviour of the (conditional) sojourn time in the M/G/1 queue with egalitarian processor sharing.

1. INTRODUCTION

In [1] we succeeded in analyzing the M/G/1 queue with egalitarian processor sharing (EPS). The tail behaviour of the sojourn time variance for a job of a given size was investigated at first in [2]. The analysis of [2] is extended in [3] where the novel result is obtained concerning an exponential asymptotics of the stationary sojourn time. The goal of this note is to present some version of this result which has a definite interest.

2. PRELIMINARIES AND THE RESULT

We consider the M/G/1—EPS queue with an arrival rate λ and processing time distribution $B(x)$ with $B(0+) = 0$ and the Laplace–Stieltjes transform (LST) $\beta(s) = \int_{0-}^{+\infty} \mathrm{e}^{-sx} dB(x)$. We require that $B(\cdot)$ has a finite first moment $\beta_1 < \infty$, and $\rho = \lambda\beta_1 < 1$. There is no overt queueing in the EPS queue because all, say n jobs present in the processor simultaneously receive service at $1/n$ times the rate given to a single job. Let$V(u)$ be the stationary sojourn time of a job whose size is u time units. We shall also assume that the moment generating function of $B(\cdot)$ exists, that is, $\beta(-s) < \infty$. In other words, $B(\cdot)$ has a light tail.

One of the direct corollaries of main theorem of [1] is the following decomposition of the random variable $V(u)$

$$V(u) \stackrel{d}{=} D(u) + \sum_{i=1}^{L} \Phi_i(u),$$

where L is the number of jobs in the system (distributed geometrically), and $\Phi_i(u)$ coincides with $\Phi(x_i, u)$ after removing the condition on the residual size $F_B = x_i$ of i–th job (that is, after averaging on $dF_B(x) = \beta_1^{-1}(1 - B(x))dx$). The random variable $\Phi(u)$ has the distribution $\Phi(x|u) = \mathsf{P}(\Phi(u) \le x|B = u)$. Besides, $D(u) \stackrel{d}{=} \Phi(x_i, u)$ for $x_i \ge u$, and $D(u)$ and $\Phi_i(u),\ i = 1, \dots, L$ are independent each from other. Then

$$\mathsf{P}\left(\sum_{i=1}^{L} \Phi_i(u) > x\right) = (1 - \rho) \sum_{n=1}^{\infty} \rho^n \left[1 - \Phi^{n*}(x|u)\right],$$

[1] This research was partially supported by Grant no. Sci.Sch.–934.2003.1 (Head R.A.Minlos) and Grant to the Program of Fundamental Researches of Russian Academy of Sciences (the Division of Informatics) “New physical and structural solutions in infotelecommunications”(Head N.A.Kuznetsov).

where the distributions of the components were deduced in [1] (in terms of the LSTs or, partially, in terms of the double LST), and $\Phi^{n*}(x|u)$ is n–fold convolution of the distribution $\Phi(x|u)$ with itself.

Now we refine the the condition $\beta(-s) < \infty$ for some $s > 0$ by the following way:
(i) there exists a constant $\theta(u) > 0$ (the Lundberg exponent) such that

$$\mathsf{E}\left[\mathrm{e}^{\theta(u)\Phi(u)}\right] = \rho^{-1}, \tag{2.1}$$

(ii) besides, there exists a constant $C(u)$ such that

$$C(u) = \rho \int_0^\infty x\mathrm{e}^{\theta(u)x} d\Phi(x|u) < \infty. \tag{2.2}$$

If the conditions (2.1) and (2.2) are valid, then the following theorem from [3] holds:

Theorem 1.

$$\mathsf{P}(V(u) > x) \sim C_1(u)\mathrm{e}^{-\theta(u)x}, \qquad x \to \infty, \tag{2.3}$$

where

$$C_1(u) = \frac{(1-\rho)\mathsf{E}\left[\mathrm{e}^{\theta(u)D(u)}\right]}{[C(u)\theta(u)]}. \tag{2.4}$$

Here $\theta(u)$ is the solution of the equation (2.1) (see the condition (i)). A sufficient condition of the existence of the solution follows from the inequality

$$\rho^{-1} < \mathsf{E}\left[\mathrm{e}^{s_1\Phi(u)}\right] < \infty.$$

If $\beta_1 = 1$, then such sufficient condition can be represented as

$$\frac{1}{\mathsf{E}\left[\mathrm{e}^{s_2(1,u)(u\wedge F_B)}\right]} < \rho.$$

□

Theorem 1 has simpler forms in the special cases (the M/H_k/1—EPS queue, the M/E_k/1—EPS queue, etc.). For example, the constant $C_1(u)$ in (2.4) reduces to the form

$$C_1(1) = [(1-\lambda)(\lambda - \theta(1)]/[2\lambda(1-\lambda) - \theta(1)\lambda(2-\lambda)]$$

in the case of the M/D/1—EPS system for which $\beta(s) = \mathrm{e}^{-su}$ and $u = 1$. Here $\theta(1)$ is unique positive solution of an equation to which the equation (2.1) is reduced:

$$\left[\lambda(\lambda - s) + s - s\mathrm{e}^{\lambda - s}\right] / \left[(\lambda - s)(\lambda - s\mathrm{e}^{\lambda - s})\right] = \lambda^{-1}$$

.

REFERENCES

1. Yashkov S.F. A derivation of response time distribution for an M/G/1 processor-sharing queue. *Problems of Control and Information Theory*, 1983, vol. 12, no. 2, pp. 133–148.
2. Yashkov S.F. A note on asymptotic estimates of the sojourn time variance in the M/G/1 queue with processor-sharing. *Systems Analysis. Modelling. Simulation*, 1986, vol. 3, no. 3, pp. 267–269.
3. Yashkov S.F. and Yashkova A.S. Asymptotics of the sojourn time in the M/GI/1 processor sharing system for the case of light tails (in Russian). *Review of the Applied and Indusrial Mathematics*, 2006, vol. 13, no. 4.

This paper was recommended for publication by V.I. Venets, a member of the Editorial Board