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A discrete-time $Geo/PH/1$ queueing system with repeated attempts [1]

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Abstract—This paper is concerned with the study of a discrete-time single-server retrial queue with geometrical interarrival times and a phase-type service process. We analyse the underlying Markov chain. An iterative algorithm to calculate the stationary distribution of Markov chain is given. By using the Fast Fourier Transform we obtain numerical examples that give the probability distribution of the number of customers in the orbit related to the specific phase in which the service process is.

1. INTRODUCTION

Retrial queueing systems arise naturally in areas such as telephone and computer communication systems. Interested readers can find comprehensive reviews in Falin [10] and Yang and Templeton [22], and a good collection on bibliographical references in [1, 2]. The early work on retrial queues was concentrated on the continuous case, but Yang and Li [21] extended the study to the discrete-time systems. Although the actual literature about continuous-time retrial queues is very large, only a few numbers of papers [3, 4, 5, 6, 9, 13, 14, 15, 18, 21] have recently appeared on discrete-time queueing systems. Indeed, the growing interest in the analysis of discrete-time retrial queues has been motivated by its applications to the field of computer and communications [8, 20].

The purpose of the present work is to analyse the $Geo/PH/1$ retrial queue. This system is a special case of the $Geo/G/1$ retrial queue studied by Yang and Li [21], but in our case the computational procedure is considerably simplified and using the methods provided by the Fast Fourier Transform (FFT) we are able to obtain numerical examples that give the probability distribution of the number of customers in the orbit related to a specific phase in the service process.

The rest of the paper is organized as follows. The next section gives a description of the mathematical model. In section 3, we provide a complete study of the Markov chain and, finally, in section 4 some numerical examples are presented.

2. THE MATHEMATICAL MODEL

We consider a discrete-time retrial queueing system in which the time axis is divided into equal intervals, called slots, and where all queueing activities (arrivals and departures) take place at the slot boundaries. For mathematical clarity, we assume that departures occur in the interval (n^-, n) and (external or repeated) arrivals occur in the interval (n, n^+), that is, departures occur at the moment immediately before the slot boundaries and arrivals occur at the moment immediately after the slot boundaries.

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External customers arrive according to a Bernoulli process with parameter a, i.e., a is the probability that an arrival takes place in a slot. If ξ is the random variable of the number of slots between two consecutive arrivals, we have

$$P[\xi = k] = \bar{a}^{k-1}\, a\,,\ k \geq 1,$$

where $\bar{a} = 1 - a$ is the probability that an arrival does not occur in a slot.

If, upon arrival, the server is idle, the service of the arriving customer begins immediately; otherwise, if the arriving customer finds the server busy, he leaves the service area and joins a group of blocked customers called *orbit* in order to get his service some time later. The time between two successive attempts by the same customer is ruled by a geometrical law with probability $1-r$, where r is the probability that a repeated customer does not make a retrial in a slot. The retrial process of a repeated customer finishes only if, upon a particular attempt, the server is idle, and the repeated customer is chosen for service among all other repeated customers who are trying for service at that time. The service time of each customer is ruled by a discrete phase-type distribution described by an irreducible PH-representation $(\vec{\beta}, B)$ of order m, where the matrix $I_m - B$ is non-singular [16]. The components of the m-dimensional column vector $\vec{\beta}$ are non-negative and satisfy the relation $\sum_{j=1}^{m} \beta_j = 1$, and the elements of the matrix B are also non-negative and verify $\sum_{j=1}^{m} B_{ij} \leq 1$ and $\sum_{j=1}^{m} B_{ij} < 1$ for at least one i, $i = 1, \ldots, m$. If the random variable τ represents the service time of a customer, we have

$$P[\tau = k] = \vec{\beta}^T\, B^{k-1}\, \vec{b}\,,\ k \geq 1$$

where $\vec{b} = (I_m - B)\,\vec{1}$ and $\vec{1}$ is a column vector of ones. The corresponding probability generating function is given by

$$P(z) = \sum_{k=1}^{\infty} P[\tau = k]\, z^k = z\,\vec{\beta}^T\, (I_m - z\,B)^{-1}\,\vec{b}$$

with factorial moments of order k:

$$M(\tau)_k = k!\,\vec{\beta}^T\, B^{k-1}\, (I_m - B)^{-k}\,\vec{1}\,,\ k \geq 1\,.$$

It is assumed that the interarrival times, the retrial times and the service times are mutually independent. To avoid trivial cases, it is also supposed $0 < a < 1$ and $0 \leq r < 1$. The traffic intensity is given by $\rho = a\, M(\tau)_1$.

3. THE MARKOV CHAIN

At time n^+, the system can be described by the Markov process $\{X_n,\, n \geq 1\}$ with $X_n = (C_n, H_n, N_n)$, where C_n denotes the server state (0 or 1 according to the server is free or busy respectively). If $C_n = 1$, H_n represents the phase of the service process at time n and N_n the number of customers in the orbit. If $C_n = 0$, H_n is not taken into consideration.

It can be shown that $\{(C_n, H_n, N_n)\ :\ n \geq 1\}$ is a Markov chain whose state space is

$$\chi = \{(0,k)\ :\ k \geq 0\,;\ (1,i,k)\ :\ i = 1, \ldots, m\,,\ k \geq 0\}.$$

Our objective is to find the stationary distribution:

$$\pi_{0,k} = \lim_{n\to\infty} P[C_n = 0\,,\ N_n = k]\,;\ k \geq 0,$$
$$\pi_{1,i,k} = \lim_{n\to\infty} P[C_n = 1\,,\ H_n = i\,,\ N_n = k]\,;\ i = 1, \ldots, m\,,\ k \geq 0.$$

The one-step transition probabilities are given by the formulae:

$$\begin{aligned}
p_{(0,k),(0,k)} &= \bar{a}\, r^k, \\
p_{(1,j,k),(0,k)} &= b_j\, \bar{a}\, r^k\,,\ j = 1, \dots, m, \\
p_{(0,k),(1,i,k)} &= a\, \beta_i, \\
p_{(0,k+1),(1,i,k)} &= \bar{a}\, (1 - r^{k+1})\, \beta_i, \\
p_{(1,j,k-1),(1,i,k)} &= B_{ji}\, a\,,\ j = 1, \dots, m\,,\ k \geq 1, \\
p_{(1,j,k),(1,i,k)} &= B_{ji}\, \bar{a} + b_j\, a\, \beta_i\,,\ j = 1, \dots, m, \\
p_{(1,j,k+1),(1,i,k)} &= b_j\, \bar{a}\, (1 - r^{k+1})\, \beta_i\,,\ j = 1, \dots, m,
\end{aligned}$$

where $\bar{a} = 1 - a$.

The system of equilibrium equations for the stationary distribution is

$$\pi_{0,k} = \bar{a}\, r^k\, \pi_{0,k} + \bar{a}\, r^k\, \vec{\pi}_{1,k}^T\, \vec{b}\,,\ k \geq 0\,, \tag{1}$$

$$\begin{aligned}
\vec{\pi}_{1,k}^T &= a\, \vec{\beta}^T\, \pi_{0,k} + \bar{a} \left(1 - r^{k+1}\right) \vec{\beta}^T\, \pi_{0,k+1} + (1 - \delta_{0,k})\, a\, \vec{\pi}_{1,k-1}^T\, B + \\
&+ \vec{\pi}_{1,k}^T\, (\bar{a}\, B + a\, \vec{b}\, \vec{\beta}^T) + \bar{a} \left(1 - r^{k+1}\right) \vec{\pi}_{1,k+1}^T\, \vec{b}\, \vec{\beta}^T\,,\ k \geq 0\,,
\end{aligned} \tag{2}$$

where $\vec{\pi}_{1,k}^T = (\pi_{1,1,k}, \pi_{1,2,k}, \dots, \pi_{1,m,k})$. The normalizing condition is:

$$\sum_{k=0}^{\infty} \pi_{0,k} + \sum_{k=0}^{\infty} \vec{\pi}_{1,k}^T\, \vec{1} = 1\,.$$

In order to solve Eqs. (1)–(2), we introduce the generating functions:

$$\varphi_0(z) = \sum_{k=0}^{\infty} \pi_{0,k}\, z^k\,,$$

$$\vec{\varphi}_1^T(z) = \sum_{k=0}^{\infty} \vec{\pi}_{1,k}^T\, z^k\,.$$

Multiplying Eqs. (1)–(2) by z^k and summing over k, these equations become:

$$\varphi_0(z) = \bar{a}\, \varphi_0(r\, z) + \bar{a}\, \vec{\varphi}_1^T(r\, z)\, \vec{b}\,, \tag{3}$$

$$\begin{aligned}
\vec{\varphi}_1^T(z) &= \frac{\bar{a} + a\, z}{z}\, \vec{\beta}^T\, \varphi_0(z) + (\bar{a} + a\, z)\, \vec{\varphi}_1^T(z)\, B + \frac{\bar{a} + a\, z}{z}\, \vec{\varphi}_1^T(z)\, \vec{b}\, \vec{\beta}^T - \\
&- \frac{\bar{a}}{z}\, \vec{\beta}^T \left[\varphi_0(r\, z) + \vec{\varphi}_1^T(r\, z)\, \vec{b}\right].
\end{aligned} \tag{4}$$

By substituting Eq. (3) into Eq. (4) we get

$$\vec{\varphi}_1^T(z) \left[I_m - (\bar{a} + a\, z) \left(B + \frac{1}{z}\, \vec{b}\, \vec{\beta}^T\right)\right] = \frac{a\, (z - 1)}{z}\, \vec{\beta}^T\, \varphi_0(z)\,. \tag{5}$$

For simplicity we introduce the following notations:

$$\begin{aligned}
C(z) &= I_m - (\bar{a} + a\, z)(B + \frac{1}{z}\, \vec{b}\, \vec{\beta}^T), \\
B(z) &= I_m - (\bar{a} + a\, z)\, B, \\
\vec{b}(z) &= \frac{\bar{a} + a\, z}{z}\, \vec{b}.
\end{aligned}$$

Therefore,

$$C(z) = B(z) - \vec{b}(z)\,\vec{\beta}^T.$$

In the next lemma we proceed to study the invertibility of the matrix $C(z)$.

Lemma 1. *If $\rho \le 1$ and $0 < z < 1$, the matrix $C(z) = B(z) - \vec{b}(z)\,\vec{\beta}^T$ is invertible and its inverse is given by:*

$$A(z) = [B(z) - \vec{b}(z)\,\vec{\beta}^T]^{-1} = [B(z)]^{-1} + \frac{[B(z)]^{-1}\,\vec{b}(z)\,\vec{\beta}^T[B(z)]^{-1}}{1-\alpha(z)},$$

where $\alpha(z) = \beta^T\,[B(z)]^{-1}\,\vec{b}(z)$.

Proof. Firstly, we point out that for $0 \le z \le 1$ is $0 \le \bar{a} + a\,z \le 1$ and consequently $[B(z)]^{-1}$ exists.

Secondly, we need to prove that the equation

$$\alpha(z) - 1 = 0 \tag{6}$$

has no solutions in $(0,1)$ if and only if $\rho \le 1$. In order to do that we observe that Eq. (6) is equivalent to equation:

$$z - P(\bar{a} + a\,z) = 0. \tag{7}$$

Let us define the functions $f_1(z) = z$ and $f_2(z) = P(\bar{a} + a\,z)$. By the convexity property of $f_2(z)$, the existence of solutions of Eq. (7) depends on the behaviour of the derivative of $f_2(z)$ in $z = 1$:

$$f_2'(1) = a\,P'(1) = a\,M(\tau)_1 = \rho\,.$$

If $\rho \le 1$, Eq. (7) does not have any solution in $(0,1)$. On the other hand, if $\rho > 1$, there exists a solution of Eq. (7) in the open interval $(0,1)$. Thus we obtain that Eq. (7) has no solution in $(0,1)$ if and only if $\rho \le 1$.

Finally, it is straigthforward to show that

$$C(z)\,A(z) = I_m.$$

Combining Eq. (5) together with Lemma 1 leads to:

$$\vec{\varphi}_1^T(z) = \frac{a\,(z-1)}{z}\,\vec{\beta}^T\,A(z)\,\varphi_0(z), \quad z \in (0,1). \tag{8}$$

Note that although the matrix $C(z)$ is not defined for $z = 0$ and $C(1)$ is not inversible, the values of $\vec{\varphi}_1^T(z)$ in $z = 0$ and $z = 1$ can be found by means of (8) by continuity.

To find out $\varphi_0(z)$ we substitute $\vec{\varphi}_1^T(r\,z)$ into Eq. (3) getting

$$\varphi_0(z) = \bar{a}\,[1 + \frac{a\,(r\,z-1)}{r\,z}\,\vec{\beta}^T\,A(r\,z)\,\vec{b}]\,\varphi_0(r\,z),$$

thats is

$$\varphi_0(z) = G(r\,z)\,\varphi_0(r\,z), \tag{9}$$

where

$$G(z) = \bar{a}\,[1 + \frac{a\,(z-1)}{z}\,\vec{\beta}^T\,A(z)\,\vec{b}].$$

Recursively applying (9) leads to

$$\varphi_0(z) = \varphi_0(0) \prod_{k=1}^{\infty} G\left(r^k z\right). \tag{10}$$

In order to find $\varphi_0(0)$ we will put $z = 1$ in (10) obtaining

$$\varphi_0(0) = \varphi_0(1) \{\prod_{k=1}^{\infty} G\left(r^k z\right)\}^{-1}. \tag{11}$$

The value of $\varphi_0(1)$ can be obtained the normalization condition:

$$\varphi_0(1) + \vec{\varphi_1}^T(1)\,\vec{1} = 1. \tag{12}$$

Multilying both sides of Eq. (8) by $\vec{1}$ and putting $z = 1$ we get

$$\vec{\varphi_1}^T(1)\,\vec{1} = a\,\lambda\,\varphi_0(1) \tag{13}$$

where

$$\lambda = \lim_{z\to 1} \vec{\beta}^T A(z)\,(z-1).$$

From (12) and (13) we have

$$\varphi_0(1) = \frac{1}{1 + a\,\lambda}. \tag{14}$$

To obtain λ we first observe using (6) that

$$\vec{\beta}^T A(z)\,\vec{1} = \frac{\vec{\beta}^T\,[B(z)]^{-1}\,\vec{1}}{1-\alpha(z)}.$$

Define $\gamma(z) = \vec{\beta}^T\,[B(z)]^{-1}\,\vec{1} = \vec{\beta}^T[I_m - (\bar{a} + a\,z)\,B]^{-1}\vec{1}$. Then we have

$$\vec{\beta}^T A(z)\,\vec{1} = \frac{\gamma(z)}{1-\alpha(z)}.$$

Therefore

$$\lambda = \lim_{z\to 1} \frac{\gamma(z)\,(z-1)}{1-\alpha(z)}.$$

Taking into account that

$$\lim_{z\to 1} \gamma(z) = E[\tau] \neq 0$$

and by applying L'Hopital rule

$$\lim_{z\to 1} \frac{z-1}{1-\alpha(z)} = \frac{1}{1-\rho}$$

we get

$$\lambda = \lim_{z\to 1} \frac{\gamma(z)\,(z-1)}{1-\alpha(z)} = \frac{E[\tau]}{1-\rho}.$$

Thus we find out the value of the unknown constant $\varphi_0(1)$:

$$\varphi_0(1) = 1 - \rho.$$

The previous results can be summarized in the following theorem:

Theorem 1. *The Markov chain $\{(C_n, H_n, N_n) \,:\, n \geq 1\}$ is ergodic if and only if $\rho \leq 1$ and its stationary distribution has the following generating functions*

$$\varphi_0(z) = (1-\rho) \prod_{k=1}^{\infty} \frac{G\left(r^k z\right)}{G\left(r^k\right)}$$

$$\vec{\varphi}_1^{\,T}(z) = \frac{a\,(z-1)}{z}\, \vec{\beta}^T A(z)\, \varphi_0(z),$$

where

$$G(z) = \bar{a}\,[1 + \frac{a\,(z-1)}{z}\, \vec{\beta}^T A(z)\, \vec{b}].$$

The stationary distribution of the server state is given by $\varphi_0(1) = 1-\rho$ and $\vec{\varphi}_1^{\,T}(1) = a\,\vec{\beta}^T (I_m - B)^{-1}$, and does not depend on the retrial parameter r.

Corollary 1. *(1) The marginal generating function of the number of customers in the orbit when the server is busy is given by:*

$$\vec{\varphi}_1^{\,T}(z)\, \vec{1} = \frac{1 - P(\bar{a} + a\,z)}{P(\bar{a} + a\,z) - z}\, \varphi_0(z)\,.$$

(2) The probability generating function of the number of customers in the orbit is given by:

$$\psi(z) = \frac{1-z}{P(\bar{a} + a\,z) - z}\, \varphi_0(z)\,.$$

(3) The probability generating function of the number of customers in the system is given by:

$$\phi(z) = \frac{(1-z)\, P(\bar{a} + a\,z)}{P(\bar{a} + a\,z) - z}\, \varphi_0(z)\,.$$

(4) The mean number of customers in the orbit is given by:

$$E[N] = \frac{a^2\, M(\tau)_2}{2\,(1-\rho)} + \sum_{k=1}^{\infty} \frac{G'\left(r^k\right)}{G\left(r^k\right)}\, r^k\,.$$

(5) The mean number of customers in the system is given by $E[L] = E[N] + \rho$.

(6) The mean time a customer spends in the system (including the service time) is given by $W = E[L]/a$.

As was expected, the results obtained in Corollary 1 agree with [21].

4. NUMERICAL WORK

This section presents some numerical results analysing the influence of the service time distribution on the stationary probabilities $\{\pi_{0,k} \,:\, k \geq 0;\ \pi_{1,i,k} \,:\, i = 1, \ldots, m,\ k \geq 0\}$. To obtain these outcomes, we carry out the numerical inversion of the generating functions $\varphi_0(z)$ and $\vec{\varphi}_1^{\,T}(z)$ using the FFT method. The FFT method is an appropriate tool to calculate the probability distribution from its generating function in case an explicit expression can be given for this generating function [19].

Throughout this section the arrival and retrial rates are assumed to be equal to $a = 0.1$ and $r = 0.6$, respectively. We present three tables, which correspond to the following PH-distributions:

$$\text{PH1:}\vec{\beta} = \begin{pmatrix} 0.3 \\ 0.1 \\ 0.6 \end{pmatrix}, \; B = \begin{pmatrix} 0.2 & 0.3 & 0.2 \\ 0.5 & 0.3 & 0 \\ 0.3 & 0 & 0.4 \end{pmatrix};$$

$$\text{PH2:}\vec{\beta} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; B = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 \end{pmatrix};$$

$$\text{PH3:}\vec{\beta} = \begin{pmatrix} 0.3 \\ 0.4 \\ 0.3 \end{pmatrix}, \; B = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.7 \end{pmatrix}.$$

Of course, the parametric values have been chosen verifying the ergodicity condition. The columns of each table provide us the probabilities of the orbit size in the different states of the server. In all the tables, we observe that the probabilities $\pi_{0,k}$ and $\pi_{1,i,k}$, $i = 1,2,3$, decrease as function of k, that is, they decrease as more customers are in the orbit. Of course, the parametric values have been chosen verifying the ergodicity condition. The columns of each table provide us the

	$\pi_{0,k}$	$\pi_{1,1,k}$	$\pi_{1,2,k}$	$\pi_{1,3,k}$
$k=0$	$6.1221 \cdot 10^{-1}$	$9.0899 \cdot 10^{-2}$	$4.5320 \cdot 10^{-2}$	$1.0563 \cdot 10^{-1}$
$k=1$	$2.6074 \cdot 10^{-2}$	$3.2666 \cdot 10^{-2}$	$1.9139 \cdot 10^{-2}$	$2.8614 \cdot 10^{-2}$
$k=2$	$3.2741 \cdot 10^{-3}$	$1.0241 \cdot 10^{-2}$	$6.3022 \cdot 10^{-3}$	$8.3280 \cdot 10^{-3}$
$k=3$	$4.9702 \cdot 10^{-4}$	$3.1076 \cdot 10^{-3}$	$1.9389 \cdot 10^{-3}$	$2.4654 \cdot 10^{-3}$
$k=4$	$8.1182 \cdot 10^{-5}$	$9.2992 \cdot 10^{-4}$	$5.8318 \cdot 10^{-4}$	$7.3071 \cdot 10^{-4}$
$k=5$	$1.3735 \cdot 10^{-5}$	$2.7638 \cdot 10^{-4}$	$1.7372 \cdot 10^{-4}$	$2.1622 \cdot 10^{-4}$
$k=6$	$2.3677 \cdot 10^{-6}$	$8.1844 \cdot 10^{-5}$	$5.1503 \cdot 10^{-5}$	$6.3881 \cdot 10^{-5}$
$k=7$	$4.1244 \cdot 10^{-7}$	$2.4186 \cdot 10^{-5}$	$1.5230 \cdot 10^{-5}$	$1.8854 \cdot 10^{-5}$
$k=8$	$7.2282 \cdot 10^{-8}$	$7.1390 \cdot 10^{-6}$	$4.4968 \cdot 10^{-6}$	$5.5610 \cdot 10^{-6}$
$k=9$	$1.2713 \cdot 10^{-8}$	$2.1057 \cdot 10^{-6}$	$1.3267 \cdot 10^{-6}$	$1.6396 \cdot 10^{-6}$
$k=10$	$2.2405 \cdot 10^{-9}$	$6.2083 \cdot 10^{-7}$	$3.9119 \cdot 10^{-7}$	$4.8328 \cdot 10^{-7}$
$k \geq 11$	$4.8018 \cdot 10^{-10}$	$2.5946 \cdot 10^{-7}$	$1.6351 \cdot 10^{-7}$	$2.0194 \cdot 10^{-7}$

Table 1. PH1.

	$\pi_{0,k}$	$\pi_{1,1,k}$	$\pi_{1,2,k}$	$\pi_{1,3,k}$
$k=0$	$3.5210 \cdot 10^{-1}$	$1.1688 \cdot 10^{-1}$	$9.5633 \cdot 10^{-2}$	$7.8245 \cdot 10^{-2}$
$k=1$	$3.7746 \cdot 10^{-2}$	$4.8839 \cdot 10^{-2}$	$5.9279 \cdot 10^{-2}$	$6.4308 \cdot 10^{-2}$
$k=2$	$7.8903 \cdot 10^{-3}$	$2.0388 \cdot 10^{-2}$	$2.6510 \cdot 10^{-2}$	$3.2925 \cdot 10^{-2}$
$k=3$	$1.7456 \cdot 10^{-3}$	$8.3303 \cdot 10^{-3}$	$1.1079 \cdot 10^{-2}$	$1.4468 \cdot 10^{-2}$
$k=4$	$3.9671 \cdot 10^{-4}$	$3.3503 \cdot 10^{-3}$	$4.5057 \cdot 10^{-3}$	$6.0089 \cdot 10^{-3}$
$k=5$	$9.1571 \cdot 10^{-5}$	$1.3349 \cdot 10^{-3}$	$1.8063 \cdot 10^{-3}$	$2.4338 \cdot 10^{-3}$
$k=6$	$2.1328 \cdot 10^{-5}$	$5.2895 \cdot 10^{-4}$	$7.1834 \cdot 10^{-4}$	$9.7320 \cdot 10^{-4}$
$k=7$	$4.9936 \cdot 10^{-6}$	$2.0894 \cdot 10^{-4}$	$2.8434 \cdot 10^{-4}$	$3.8642 \cdot 10^{-4}$
$k=8$	$1.1727 \cdot 10^{-6}$	$8.2378 \cdot 10^{-5}$	$1.1224 \cdot 10^{-4}$	$1.5281 \cdot 10^{-4}$
$k=9$	$2.7592 \cdot 10^{-7}$	$3.2443 \cdot 10^{-5}$	$4.4237 \cdot 10^{-5}$	$6.0290 \cdot 10^{-5}$
$k=10$	$6.4986 \cdot 10^{-8}$	$1.2768 \cdot 10^{-5}$	$1.7418 \cdot 10^{-5}$	$2.3753 \cdot 10^{-5}$
$k \geq 11$	$2.0041 \cdot 10^{-8}$	$8.2792 \cdot 10^{-6}$	$1.1298 \cdot 10^{-5}$	$1.5415 \cdot 10^{-5}$

Table 2. PH2.

	$\pi_{0,k}$	$\pi_{1,1,k}$	$\pi_{1,2,k}$	$\pi_{1,3,k}$
$k=0$	$7.6711 \cdot 10^{-1}$	$3.4665 \cdot 10^{-2}$	$6.8910 \cdot 10^{-2}$	$7.6825 \cdot 10^{-2}$
$k=1$	$1.4275 \cdot 10^{-2}$	$2.4686 \cdot 10^{-3}$	$9.4912 \cdot 10^{-3}$	$1.8132 \cdot 10^{-2}$
$k=2$	$1.0086 \cdot 10^{-3}$	$3.0252 \cdot 10^{-4}$	$1.3445 \cdot 10^{-3}$	$3.9673 \cdot 10^{-3}$
$k=3$	$9.6701 \cdot 10^{-5}$	$5.1523 \cdot 10^{-5}$	$2.0998 \cdot 10^{-4}$	$8.4841 \cdot 10^{-4}$
$k=4$	$1.0559 \cdot 10^{-5}$	$9.9065 \cdot 10^{-6}$	$3.6284 \cdot 10^{-5}$	$1.7968 \cdot 10^{-4}$
$k=5$	$1.2302 \cdot 10^{-6}$	$1.9933 \cdot 10^{-6}$	$6.7807 \cdot 10^{-6}$	$3.7876 \cdot 10^{-5}$
$k=6$	$1.4830 \cdot 10^{-7}$	$4.0918 \cdot 10^{-7}$	$1.3332 \cdot 10^{-6}$	$7.9647 \cdot 10^{-6}$
$k=7$	$1.8212 \cdot 10^{-8}$	$8.4824 \cdot 10^{-8}$	$2.6998 \cdot 10^{-7}$	$1.6727 \cdot 10^{-6}$
$k=8$	$2.2598 \cdot 10^{-9}$	$1.7674 \cdot 10^{-8}$	$5.5565 \cdot 10^{-8}$	$3.5104 \cdot 10^{-7}$
$k=9$	$2.8204 \cdot 10^{-10}$	$3.6927 \cdot 10^{-9}$	$1.1535 \cdot 10^{-8}$	$7.3642 \cdot 10^{-8}$
$k=10$	$3.5318 \cdot 10^{-11}$	$7.7271 \cdot 10^{-10}$	$2.4057 \cdot 10^{-9}$	$1.5445 \cdot 10^{-8}$
$k \geq 11$	$5.0666 \cdot 10^{-12}$	$2.0472 \cdot 10^{-10}$	$6.3610 \cdot 10^{-10}$	$4.0983 \cdot 10^{-9}$

Table 3. PH3.

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