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Effecivity properties of intuitionistic set theory

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Abstract—Let us consider two-sorted intuitionistic set theory ZFI2 with sort 0 for natural numbers and sort 1 for sets. We shall show that well-known Church rule with parameters of sort 1 is admissible in ZFI2 in some rather strong sense, and get from this point the admissibility of Markov rule with all parameters in ZFI2, and also DP and numerical EP with set parameters for it in the same sense.

1. INTRODUCTION

Let $\mathbb{ZFI}2$ be usual first order intuitionistic Zermelo-Fraenkel set theory in two-sorted language (where sort 0 - for natural numbers, and sort 1 - for sets) which contains functional symbols for all primitive recursive functions (p.r.f.), and predicate symbols $=^0$ for equality of natural numbers, $\in^0$ for membership of natural to set and $\in^1$ for membership of set to set. Axioms and rules of this system are: all usual axioms and rules of intuitionistic predicate logic ($\mathbb{HPC}$), all usual axioms of intuitionistic arithmetic ($\mathbb{HA}$) for variables of sort 0, and all usual axioms and schemes of Zermelo-Fraenkel system for variables of sort 1, namely axioms Extensionality (Ext), Pair ($Pair$), Union (Un), Infinity (Inf), Power set (Pow), and schemes Separation (Sep), Transfinite Induction (TI) as Regularity and Collection ($Coll$) as Substitution.

Formal notation of all schemes and axioms is set below, in section 2.3.

It is well-known that $\mathbb{ZFI}2$ have important effectivity properties: disjunction property (DP), numerical existence property(EP),and also that the Markov rule is admissible in it. Such collection of properties shows that it is a sufficiently constructive theory.

On the other hand, a lot of usual informal mathematical reasons may be formalized in it, so, we can formalize in $\mathbb{ZFI}2$ and decide a lot of informal problems about transformation of some classical proof into intuitionistical, and extraction of some description of a mathematical object from some proof of its existence. The well-known example of result of this kind is the well-known theorem of Lyubetsky [5]. The below problems were formulated and discussed as hypotheses in lecture courses given by V. Lyubetsky on Intuitionistic set theory at Moscow State University in spring term of 1995.

In this paper we prove the admissibility of Curch rule (CR)in ZFI2 with set parameters, and show how the admissibility of Markov rule (MR) with all parameters can be extracted from it.

Short review

The section 1 contains some necessary introduction.

In section 2 we introduce formalized realizability $e\,\mathrm{Q}\,\varphi$ which generalizes on the set theoretical level well-known q-realizability from Friedman [2], after that we consider the theory $\mathbb{ZFI}2^c$ which is obtained from $\mathbb{ZFI}2$ by adding of countable set of new constants of sort 1, prove the correctness of this theory w.r.t. our realizabity.

In section 3 we get from this theorem the following properties of effectivity for $\mathbb{ZFI}2$:

1. Disjunction property (DP) with parameters of sort 1;
2. Numerical existence property with parameters of sort 1;
3. Admissibility of Church rule with parameters of sort 1;
4. Admissibility of Markov rule with parameters of any sorts.

So, we can conclude that $ZFI2$ is an effective theory wich formalizes "effective mathematics."

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2. THE Q-REALIZABILITY FOR SET THEORY

We shall use all usual set-theoretical notations (see f.e. [1]), and elemetary recursion theory fomalized in $\mathbb{HA}$ (see [3]).

2.1. Preliminaries

On definable constants

We can note at beginnig that it can add to language of theory $\mathbb{ZFI}2$ each definable constant. More exactly, let T is a theory in our language, and the theory T^* is obtained by adding to T some set of definable constants. Then if Church rule is admissible in T^* (with or without parameters) then it is admsissible in T, too.

On free constants

Now we add to language of theory $\mathbb{ZFI}2$ a countable set of new constants of sort 1 ("free constants"). Let, again, T be a theory in the language of $\mathbb{ZFI}2$, and let the theory T^c be obtained from T by adding of all these free constants. It is well-know that T^c is conservative over T.

Let T be a simple extension of $\mathbb{ZFI}2$. We claim that if Church rule without parameters is admisible in T^c then Church rule with parameters of sort 1 is admissible in T. Indeed, let φ be some formula of the language of T^c without number parameters. Let also $T \vdash \forall x^0 \exists y^0 \varphi(x; y; \overrightarrow{p})$, where $\overrightarrow{p}$ is a list of all parameters of this formula. Take a proof of $\forall x^0 \exists y^0 \varphi(x; y \overrightarrow{p})$ in T and replace in this proof all parameters from $\overrightarrow{p}$ to free constasnts $c_1, \ldots, c_n$. Clearly, it obtains a new proof in T^c of the formula $\forall x^0 \exists y^0 \varphi(x; y; \overrightarrow{c})$, where $\overrightarrow{c}$ is a list of all used new constants. This formula already does not contain any parameters, and admissibility of Church rule without parameters for T^c gives us the natural number ρ such that we have:

$$T \vdash \forall x^0 \exists y^0 [y = \{\rho\}(x). \wedge .\varphi(x; y; \overrightarrow{c})].$$

Let us take any proof of this formula and replace each free constant to new parameter. Clearly, we obtain a proof of the formula

$$T \vdash \forall x^0 \exists y^0 [y = \{\rho\}(x). \wedge .\varphi(x; y; \overrightarrow{p})]$$

in the theory T. Of course, this remark have a completely general kind. We shall use it below to demonstrate also the admissibility of Markov rule with all parameters. We could use it to redemonstrate for $\mathbb{ZFI}2^-$ DP and EP with set parameters and also the uniformization rule (UR) with set parameters for this theory. (See our next paper about that).

2.2. Universum Δ

In this subsection we define universum Δ and prove some its useful properties.

Definition 1. The Von Neumann universum Π is defined as usually:

$$\Pi = \bigcup_{\alpha\in\mathbf{On}} V_\alpha, \text{ where}$$

$$V_\alpha = \omega \cup \bigcup_{\beta<\alpha} \mathcal{P}(V_\beta).$$

Definition 2. Let us define the relation $y \in^* x$ as $(\exists n \in \omega)[\langle n, y\rangle \in x]$. Also we define "the *-rank of set x" as an ordinal $\mathrm{rk}^*(x) \leftrightharpoons \bigcup\{\mathrm{rk}^*(y)+1 | y \in^* x\}$.

Remark 1. If $y \in^* x$ then $\mathrm{rk}^*(y) < \mathrm{rk}^*(x)$.

Definition 3. Let us define surjection $\mathfrak{F} : x \mapsto \overset{\circ}{x}$ from class **V** onto class Δ. It is an identity function on naturals, and on sets we define:

$$\mathfrak{F}(x) = \overset{\circ}{x} = \{\langle n, \overset{\circ}{y}\rangle | (\langle n, y\rangle \in x) \wedge (x \in \Delta_\alpha)\} \cup \{\langle n, d\rangle | (\langle n, d\rangle \in x) \wedge (x \in \Delta_\alpha)\},$$

where $\alpha = \mathrm{rk}^*(x)$.

Definition 4. Now let us define the relations $x\,\mathrm{ext}_\alpha\, h$ (where x is a set, h is a natural, α is a ordinal), $y \overset{\overrightarrow{h}}{\sim}_\gamma z$, where $\overrightarrow{h}$ is a quadruple $\langle h_1, h_2, h_3, h_4\rangle$ of natural numbers, sets Δ_α and universum Δ by transfinite induction as follows:

$$x\,\mathrm{ext}_\alpha\, h \leftrightharpoons (\exists\beta < \alpha) x\,\mathrm{ext}_\beta\, h \vee [x \subseteq (\omega\times\omega) \cup \bigcup_{\beta<\alpha} (\omega \times \Delta_\beta) | (x \text{ satisfies } (*))],$$

where (*) is the following condition for x: for all ordinals $\gamma < \alpha$, for all $\overrightarrow{h} = \langle h_1; h_2; h_3; h_4\rangle$, where $h_i \in \omega$, for each natural n, and for all sets y, z

$$\text{if } \; y \underset{\gamma}{\overset{\overrightarrow{h}}{\sim}} z \; \text{ and } \; \langle n; \overset{\circ}{y}\rangle \in \overset{\circ}{x} \; \text{ then } \; !\{h\}(n, \overrightarrow{h}) \; \textit{and} \; \langle \{h\}(n, \overrightarrow{h}); \overset{\circ}{z}\rangle \in \overset{\circ}{x}\,.$$

Now let us define the relation of "effective equality of two sets y and z by the function h":

$$y \underset{\gamma}{\overset{\overrightarrow{h}}{\sim}} z \leftrightharpoons (\exists\beta < \alpha)[y \underset{\gamma}{\overset{\overrightarrow{h}}{\sim}} z] \vee [(y, z \in \Delta_\alpha) \wedge (\forall n, k, h, u)[(1) \wedge (2) \wedge (*)]], \; \textit{where}$$

(1) $\langle n, k\rangle \in \overset{\circ}{y} \wedge (k \in y) \to !\{h_1\}(n; k) \wedge \langle \{h_1\}(n; k); k\rangle \in \overset{\circ}{z}$;
(2) $\langle n, k\rangle \in \overset{\circ}{z} \wedge (k \in z) \to !\{h_2\}(n; k) \wedge \langle \{h_2\}(n; k); k\rangle \in \overset{\circ}{y}$;
(*) if for all $\beta < \alpha$ *we have* $\overset{\circ}{u}\,\mathrm{ext}_\beta\, h$ then
(3) $\langle n, \overset{\circ}{u}\rangle \in \overset{\circ}{y} \wedge (u \in y) \to !\{h_3\}(n; \overrightarrow{h}) \wedge \langle \{h_3\}(n; \overrightarrow{h}); \overset{\circ}{u}\rangle \in \overset{\circ}{z}$;
(4) $\langle n, \overset{\circ}{u}\rangle \in \overset{\circ}{z} \wedge (u \in z) \to !\{h_4\}(n; \overrightarrow{h}) \wedge \langle \{h_4\}(n; \overrightarrow{h}); \overset{\circ}{u}\rangle \in \overset{\circ}{y}$.

After that we can define:

$$y \underset{\gamma}{\sim} z \leftrightharpoons (\exists y)[y \underset{\gamma}{\overset{\overrightarrow{h}}{\sim}} z], \textit{and} \;\; y \sim z \leftrightharpoons (\exists\gamma)[y \underset{\gamma}{\sim} z].$$

Finitely, we can define:

$$\Delta_\alpha = \{x | (\exists h \in \omega)[\, x\,\mathrm{ext}_\alpha\, h\,]\}, \; \textit{and}$$

$$\Delta = \bigcup_\alpha \Delta_\alpha.$$

Lemma 1 (Properties of the function $\mathfrak{F} : x \mapsto \overset{\circ}{x}$).

(i) *For each set* x *we have:* $\overset{\circ}{x} \in \Delta$
(ii) *If* $x \in \Delta$ *then* $\overset{\circ}{x} = x$.
(iii) *If* $x \notin \Delta$ *then* $\overset{\circ}{x} = \emptyset$.
(iv) $\mathfrak{F} : x \mapsto \overset{\circ}{x}$ *is a surjection of* Π *onto* Δ.

Lemma 2 (Properties of the class Δ). *In* $\mathbb{ZFI}2$ *the following properties are provable:*

(i) *The class* Δ_α *is a set.*
(ii) *For each ordinals* $\alpha < \beta$ *we have* $\Delta_\alpha \subset \Delta_\beta$.
(iii) *For all sets* $x, y \in \Delta$ *and* $\alpha \in \mathbf{On}$ *we have*

$$(y \in^* x) \wedge (x \in \Delta_\alpha) \rightarrow (y \in^1 \Delta_\alpha).$$

(iv) *for each set* $x \in \Delta$ *we have:* $x \in \Delta_{\mathrm{rk}^*(x)+1}$.

2.3. Definition of realizability

Recall that there is well known bijection $j : \omega \times \omega \rightarrow \omega$ as well two projections $j_1, j_2 : \omega \times \omega \rightarrow \omega$ such that in $\mathbb{HA}$ we have $j(j_1 u^0; j_2 u^0) = u^0$.

Now let us define our realizability by induction on comlexity of formula φ.

Definition 5. Namely, for each formula φ we define a new formula $e\,\mathrm{Q}\,\varphi$ with parameter e (of sort 0) as well all parameters of formula φ.

1. If φ is $t =^0 s$, then $e\,\mathrm{Q}\,\varphi$ is $t =^0 s$.
2. If φ is $t \in^0$ x, where t is a valuated term of sort 0 and x is a set, then $e\,\mathrm{Q}\,\varphi$ is $\langle e; t\rangle \in \overset{\circ}{x}$.
3. If φ is $x \in^1 y$, where x and y are sets, then $e\,\mathrm{Q}\,\varphi$ is $\langle e; \overset{\circ}{x}\rangle \in \overset{\circ}{y}$.
4. If φ is $\psi \wedge \vartheta$, then $e\,\mathrm{Q}\,\varphi$ is $j_1 e\,\mathrm{Q}\,\psi. \wedge .j_2 e\,\mathrm{Q}\,\vartheta$.
5. If φ is $\psi \vee \vartheta$, then $e\,\mathrm{Q}\,\varphi$ is

$$[j_1 e = 0 \rightarrow j_2 e\,\mathrm{Q}\,\psi. \wedge .\psi]. \wedge .[j_1 \neq 0 \rightarrow j_2 e\,\mathrm{Q}\,\vartheta. \wedge .\vartheta].$$

6. If φ is $\psi \rightarrow \vartheta$, then $e\,\mathrm{Q}\,\varphi$ is $\forall p[p\,\mathrm{Q}\,\psi. \wedge .\psi \rightarrow !\{e\}(p). \wedge .\{e\}(p)\,\mathrm{Q}\,\vartheta]$.
7. If φ is $\forall u^0 \psi$, then $e\,\mathrm{Q}\,\varphi$ is $\forall d[!\{e\}(d). \wedge .\{e\}(d)\,\mathrm{Q}\,\psi(d)]$.
8. If φ is $\exists u^0 \psi$, then $e\,\mathrm{Q}\,\varphi$ is $j_2 e\,\mathrm{Q}\,\psi(j_1 e). \wedge .\psi(j_1 e)$.
9. If φ is $\forall x^1 \psi$, then $e\,\mathrm{Q}\,\varphi$ is $\forall x^1 \forall h[\overset{\circ}{x} .ext.h \rightarrow !\{e\}(h) \wedge \{e\}(h)\,\mathrm{Q}\,\psi(x)]$.
10. If φ is $\exists x^1 \psi$, then $e\,\mathrm{Q}\,\varphi$ is $\exists x^1[\overset{\circ}{x} .ext.j_2 e \wedge j_1 e\,\mathrm{Q}\,\psi(x)]$.

2.4. Correctness theorem

Theorem 1. *If* $\mathbb{ZFI}2^c \vdash \varphi$*, where* φ *is a closed formula then for each derivation of* φ *in theory* $\mathbb{ZFI}2^c$ *there exists some effectively found number* e *such that* $\mathbb{ZFI}2^c \vdash e\,\mathrm{Q}\,\varphi$.

Proof. We only check four more difficult schemes: TI, $Coll$, Sep, and Ext.

1. Extensionality Axiom (Ext)

$$(\forall x^1 \forall y^1 \forall z^1)[(\forall a^0)(a \in x \equiv a \in y) \wedge (\forall v^1)((v \in x \equiv v \in y))] \wedge (x \in z) \rightarrow (y \in z).$$

Let us find a natural e such that $\mathbb{ZFI}2 \vdash e\,\mathrm{Q}\,Ext$. Reason in $\mathbb{ZFI}2$. We need a natural e such that we woud have: for all x^1,h_x, y^1,h_y,z^1, and h_z if $\overset{\circ}{x}\,.ext.h_x$, $\overset{\circ}{y}\,.ext.h_y$, and $\overset{\circ}{z}\,.ext.h_z$ then the number $e^* := \{\{\{e\}(h_x)\}(h_y)\}(h_z)$ is defined and realizes the formula

$$[(\forall a^0)(a \in x \equiv a \in y) \wedge (\forall v^1)((v \in x \equiv v \in y))] \wedge (x \in z) \to (y \in z).$$

This means that if some natural p realizes the antecedent i.e. the formula

$$[(\forall a^0)(a \in x \equiv a \in y) \wedge (\forall v^1)((v \in x \equiv v \in y))] \wedge (x \in z),$$

and this formula is true then $\{e^*\}(p)$ is defined and realizes the consequent, i.e. $\langle \{e^*\}(p); \overset{\circ}{y}\rangle \in \overset{\circ}{z}$. Let p realizes the antecedent of Ext. It means that $p = \nu(p_1, p_2, p_3)$ and the following conditions must be satisfied:

(a) $p_1\,\mathrm{Q}[(\forall a^0)(a \in x \equiv a \in y)]$, i.e. for all natural q the number $\{p_1\}(q)$ is defined and realizes the formula $(q \in x \equiv q \in y)$, i.e. $\{p_1\}(q) = j(p_{11}; p_{12})$ and $p_{11}\,\mathrm{Q}[a \in x \to a \in y]$, and $p_{12}\,\mathrm{Q}(a \in y \to a \in x)$. So, the condition (a) means:

(a1) for each natural number r if $\langle r; q\rangle \in \overset{\circ}{x}$ and $q \in^0 x$ then $\{\{p_{11}\}(q)\}(r)$ is defined and

$$\langle\ \{\{p_{11}\}(q)\}(r); q\rangle \in \overset{\circ}{y}\ .$$

(a2) for each natural number s if $\langle s; q\rangle \in \overset{\circ}{y}$ and $q \in^0 x$ then $\{\{p_{12}\}(q)\}(s)$ is defined and

$$\langle\ \{\{p_{12}\}(q)\}(s); q\rangle \in \overset{\circ}{x}\ .$$

(b) $p_2\,\mathrm{Q}[\forall v^1)(v \in x \equiv v \in y)]$, i.e. for each set v, and for each natural h_v if $v.ext.h_v$ then $\{p_2\}(h_v)$ is defined and realizes the formula $(v \in x \equiv v \in y)$, i.e. $\{p_2\}(h_v) = j(p_{21}; p_{22})$, and $p_{21}\,\mathrm{Q}(v \in x \to v \in y)$, and $p_{22}\,\mathrm{Q}(v \in y \to v \in x)$.

So, the condition (b) means: for each set v, for each h_v if $v.ext.h_v$ then

(b1) for each natural q if $\langle q; \overset{\circ}{v}\rangle \in \overset{\circ}{x}$ and $v \in^1 x$ then $\{p_{21}\}(q)$ is defined and $\langle \{p_{21}\}(q); \overset{\circ}{v}\rangle \in \overset{\circ}{y}$.

(b2) for each natural number q if $\langle q; \overset{\circ}{v}\rangle \in \overset{\circ}{y}$ and $v \in^1 y$ then $\{p_{22}\}(q)$ is defined and $\langle \{p_{22}\}(q); \overset{\circ}{v}\rangle \in \overset{\circ}{x}$.

(c) $\langle p_3; \overset{\circ}{x}\rangle \in \overset{\circ}{z}$. So, by our hypothesys, the conditions (a)-(c) are satisfied for our number p. Then we have to find a natural number $\{e^*\}(p)$ such that $\langle \{e^*\}(p); \overset{\circ}{y}\rangle \in \overset{\circ}{z}$. Let us define:

$$\{h_1\}(n, k) := \{j_1(\{p_1\}(k))\}(n);$$

$$\{h_2\}(n, k) := \{j_2(\{p_1\}(k))\}(n);$$

$$\{h_3\}(n, k) := \{j_1(\{p_2\}(k))\}(n);$$

$$\{h_4\}(n, k) := \{j_2(\{p_2\}(k))\}(n);$$

$$\overrightarrow{h} = \langle h_1, h_2, h_3, h_4\rangle.$$

Let also $\mathrm{rk}^*(x) = \alpha$, and $\mathrm{rk}^*(z) = \beta$. Since $\langle p_3; \overset{\circ}{x}\rangle \in \overset{\circ}{z}$, we have: $\alpha < \beta$. Let also again: $x^* := x_1^* \cup x_2^*$, where we define: $x_1^* := \{z | (\overset{\circ}{z} \in^* \overset{\circ}{x}) \wedge z \in x\}$, $x_2^* := \{k \in \omega | (k \in^* \overset{\circ}{x}) \wedge k \in^0 x\}$. So, we have by the conditions (a)-(b): $x^* = y^*$. Therefore, $\mathrm{rk}^*(x) = \mathrm{rk}^*(y) = \alpha$. So,we have:

$$x \underset{\alpha}{\overset{\overrightarrow{h}}{\sim}} y, \quad and \quad \alpha < \beta.$$

But $z.ext.h_z$, and $\mathrm{rk}(z) = \beta$, so,the number $\{h_3\}(p_3; h_z)$ is defined and $\langle\{h_3\}(p_3; h_z); \overset{\circ}{y}\rangle \in \overset{\circ}{z}$. Let $\{e^*\}(p) := \{h_3\}(p_3; h_z)$. Then e^* realizes Ext. Indeed, if p realizes the antecedent of Ext then $p = \nu(p_1, p_2, p_3)$, where these p_i satisfies the conditions (a)-(c). Then if $x.ext.h_x$, $y.ext.h_y$, and $z.ext.h_z$ then $\{h_3\}(p_3; h_z)$ is defined and $\langle\{h_3\}(p_3; h_z); \overset{\circ}{y}\rangle \in \overset{\circ}{z}$, i.e. we have: $\{e^*\}(p)$ is defined and $\langle\{e^*\}(p); \overset{\circ}{y}\rangle \in \overset{\circ}{z}$. The existence of such number e^* it can be got from (formalized)recursion theorem: for partially recursive term $t(h_x, h_y, h_z) \leftrightharpoons \{h_3\}(p; I^3_3(h_x; h_y; h_z))$ there is a primitive recursive term $t_1(h_x; h_y; h_z)$ such that $t_1(h_x; h_y; h_z) \sim \{h_3\}(p; I^3_3(h_x; h_y; h_z)) \sim \{h_3\}(p\ ; h_z))$. There is a number e such that $\{e\}(h_x; h_y; h_z) \sim t_1(h_x; h_y; h_z)$. We have (even in HA):

$$\{\{e\}(h_x; h_y; h_z)\}(p) \sim \{h_3\}(p\ ; h_z).$$

2. Transfinite Induction (TI)

$$\forall x^1[(\forall y \in^1 x)\varphi(y) \to \varphi(x)] \to \forall x^1\varphi(x).$$

Let us find a natural number e such that $e\,\mathrm{Q}\,TI$, i.e. for each natural p if $(\forall x^1)[(\forall y \in^1 x)\varphi(y)]$ and $p\,\mathrm{Q}\,\forall x^1[(\forall y \in^1 x)\varphi(y) \to \varphi(x)]$ then $\{e\}(p)$ is defined and $\{e\}(p)\,\mathrm{Q}\,\forall x^1\varphi(x)$. By our definition of "$e\,\mathrm{Q}\,\varphi$" this signifies the following: if the following conditions are satisfied

(i) $(\forall x^1)(\forall h_x \in \omega)[\overset{\circ}{x}\,.ext.h_x \to \{p\}(h_x)\,\mathrm{Q}((\forall y \in^1 x)\varphi(y) \to \varphi(x))]$, and
(ii) $(\forall x^1)[(\forall y \in^1 x)\varphi(y) \to \varphi(x)]$,

then $!\{e\}(p)$ and for all set x and number h_x if $\overset{\circ}{x}\,.ext.h_x$ then $\{e\}(p)\,\mathrm{Q}\,\varphi(x)$.

Let us fix a number p such that the conditions (i) and (ii) are satisfied, and an arbitrary $x \in^1 \Pi$, and some natural number h_x , such that $\overset{\circ}{x}\,.ext.h_x$.

The condition (i) means: for each natural number q if $q\,\mathrm{Q}(\forall y \in^1 x)\varphi(y)$ and $(\forall y \in^1 x)\varphi(y)$ then $!\{p\}(q)$ and $\{p\}(q)\,\mathrm{Q}\,\varphi(x)$. In the other words, this means that if $(\forall y \in^1 x)\varphi(y)$, and if for each set y, and for each number h_y, such that $\overset{\circ}{y}\,.ext.h_y$, we have: $!\{q\}(h_y)$, and $\{q\}(h_y)\,\mathrm{Q}[y \in^1 x \to \varphi(y)]$ then $!\{p\}(q)$ and $\{p\}(q)\,\mathrm{Q}\,\varphi(x)$. Finitely, the condition (i) means: if for each set y and naturals h_y and s such that $\overset{\circ}{y}\,.ext.h_y$, we have: $\langle s; \overset{\circ}{y}\rangle \in\overset{\circ}{x}$ and $y \in^1 x$ implies $!\{\{q\}(h_y)\}(s)$ and $\{\{q\}(h_y)\}(s)\,\mathrm{Q}\,\varphi(y)$ then $!\{p\}(q)$ and $\{p\}(q)\,\mathrm{Q}\,\varphi(x)$.

The condition (ii) means: for each $x^1 \in \Pi$ we have: $(\forall y \in^1 x)\varphi(y)$.

Let

$$\{I(p)\}(h) \simeq \{\{e\}(p)\}(h) \simeq \{\{p\}(h)\}(u),$$

where $\{\{u\}(h)\}(s) = \{I(p)\}(h)$.

It exists by the recursion theorem. If now we prove that $u\,\mathrm{Q}(\forall y \in^1 x)\varphi(y)$ then we will get by the choose of the number p that $\{\{p\}(h_x)\}(u)$, is defined, and $\{\{p\}(h_x)\}(u)\,\mathrm{Q}\,\varphi(x)$, i.e. $\{I(p)\}(h_x)\,\mathrm{Q}\,\varphi(x)$. So, we have to show that for each $y \in^1 x$ for each number h_y such that $\overset{\circ}{y}\,.ext.h_y$ we have:

$$\{u\}(h_y)\,\mathrm{Q}(y \in^1 x \to \varphi(y)),$$

i.e. for each natural s

$$\text{if } \langle s; \overset{\circ}{y}\rangle \in\overset{\circ}{x} \text{ and } y \in^1 x \text{ then } !\{\{u\}(h_y)\}(s) \text{ and } \{\{u\}(h_y)\}(s)\,\mathrm{Q}\,\varphi(y).$$

But if $y \in^1 x$ then $\mathrm{rk}(y) < \mathrm{rk}(x)$, and by IH we have $\{I(p)\}(h_y)\,\mathrm{Q}\,\varphi(y)$, so $\{\{u\}(h_y)\}(s)\,\mathrm{Q}\,\varphi(y)$ by the definition of the number u. Thus, $u\,\mathrm{Q}(\forall y \in^1 x)\varphi(y)$, and consequently, $!\{\{p\}(h_x)\}(u)$ and $\{\{p\}(h_x)\}(u)\,\mathrm{Q}\,\varphi(x)$, i.e. $\{I(p)\}(h_x)\,\mathrm{Q}\,\varphi(x)$. So, $I(p)\,\mathrm{Q}\,\forall x^1\varphi(x)$.

$$3.\ \ Coll \qquad \forall z^1[z \in^1 a \to \exists y^1\ \psi(z;y)] \wedge \forall v^0[v \in^0 a \to \exists y^1\ \varphi(v;y)] \to$$
$$\to \exists x^1[(\forall z \in^1 a)(\exists y^1 \in x)\psi(z;y) \wedge (\forall v^0 \in^0 a)(\exists y^1 \in x)\varphi(v;y).]$$

Let us find a natural number e such that $e\,\mathrm{Q}\,Coll$. This means that if p realizes the antecedent of $Coll$ then $\{e\}(p)$ realizes it's consequent. Reason in our theory: fix any $a \in^1 \Pi$. Then if the following conditions are satisfied:

1. $j_1p\,\mathrm{Q}(\forall z \in^1 a)(\exists y^1)\psi(z;y)$;
2. $j_2p\,\mathrm{Q}(\forall v \in^0 a)(\exists y^1)\varphi(v;y)$;
3. $(\forall z \in^1 a)(\exists y^1)\psi(z;y)$;
4. $(\forall v \in^0 a)(\exists y^1)\varphi(v;y)$;

then $\{e\}(p)$ is defined and realizes the consequent of $Coll$.

The condition 1 means: for each set z and $h_z \in \omega$ if $\overset{\circ}{z}.ext.h_z$, then for each natural d if $\langle d;\overset{\circ}{z}\rangle \in\overset{\circ}{a}$ and $z \in^1 a$ then $!\{\{j_1p\}(h_z)\}(d)$ and for some y^1 we have: $\{\{j_1p\}(h_z)\}(d) = j(d_1;d_2)$, and $\overset{\circ}{y}.ext.d_2$, and $d_1\,\mathrm{Q}\,\psi(z;y)$.

The condition 2 means: for each naturals q and d we have: $!\{j_2p\}(d)$ and if $\langle q;d\rangle \in\overset{\circ}{a}$ and $d \in^0 a$ then $\{\{j_2p\}(d)\}(q)$ is defined and realizes the formula $\varphi(d;y)$ for some y^1.

The conclusion of the formula $e\,\mathrm{Q}\,Coll$ means: $\{e\}(p) = j(e_1;e_2)$ and there is some set x such that:

(i) $\overset{\circ}{x}.ext.e_2$, and

(ii) $e_1 = j(e_{11};e_{12})$

(iii) $e_{11}\,\mathrm{Q}\,\forall z^1(\dots)$, i.e. for each set z and each natural h_z if $z.ext.h_z$ then $!\{e_{11}\}(h_z)$, and for each natural d if $\langle d;\overset{\circ}{z}\rangle \in\overset{\circ}{a}$ and $z \in^1 a$ then $!\{\{e_{11}\}(h_z)\}(d)$, $\{\{e_{11}\}(h_z)\}(d) = j(d_1;d_2)$, and also $d_1 = j(d_{11};d_{12})$, and for some set y we have: $\overset{\circ}{y}.ext.d_2$, and $\langle d_{11};\overset{\circ}{y}\rangle \in\overset{\circ}{x}$, and $d_{12}\,\mathrm{Q}\,\psi(z;y)$.

(iv) For each natural d it satisfies: $!\{e_2\}(d)$ and for each natural q we have: if $\langle q;d\rangle \in\overset{\circ}{a}$ and $d \in^0 a$ then $!\{\{e_2\}(d)\}(q)$ and $\langle j_1\{\{e_2\}(d)\}(q);d\rangle \in\overset{\circ}{x}$, and $j_1\{\{e_2\}(d)\}(q)\,\mathrm{Q}\,\varphi(d;y)$.

By the schema Collection it follows from conditions 1 and 2 that there is some set S such that the following conditions are satisfied:

- for each natural d and set z, and h_z if $\overset{\circ}{z}.ext.h_z$ then if $\langle d;\overset{\circ}{z}\rangle \in\overset{\circ}{a}$ and $z \in a$ then $\{\{j_1p\}(h_z)\}(d)$ is defined and for some $y \in^1 S$ it satisfies: $\{\{j_1p\}(h_z)\}(d)\,\mathrm{Q}\,\psi(z;y)$
- for each natural d we have: $!\{j_2p\}(d)$, and if for some natural s $\langle q;s\rangle \in\overset{\circ}{a}$ and $s \in^0 a$ then $\{\{j_2p\}(d)\}(q)$ is defined and realize the formula $\varphi(d;y)$ for some $y^1 \in S$. Let $x_0 \leftrightharpoons \omega\times\{\overset{\circ}{y}\,|y \in^1 S\}$. Clearly, $x_0 \in \Delta$, so, $\overset{\circ}{x}_0 = x_0$.

Let us define functions $\{e_1\}$ and $\{e_2\}$.

* If for some natural d and z^1 it satisfies: $\langle d;\overset{\circ}{z}\rangle \in\overset{\circ}{a}$, and $z \in^1 a$ then $!\{\{j_1p\}(h_z)\}(d)$, and for some y^1 we have: $\{\{j_1p\}(h_z)\}(d)\,\mathrm{Q}\,\psi(z;y)$.
Now let us define: $\{\{j_1p\}(h_z)\}(d) = d$, and $j_2\{e_1\}(d) = \{j_2p\}(d)$, i.e. $\{e_1\}(d) = j(d;\{j_2p\}(d)$.
In this case we have: $\langle j_1\{e_1\}(d);\overset{\circ}{y}\rangle \in\overset{\circ}{x}$, and $j_2\{e_1\}(d)\,\mathrm{Q}\,\psi(y;z)$, i.e. $\{e_1\}(d) = j(d_1,d_2)$, and $\langle d_1;\overset{\circ}{y}\rangle \in\overset{\circ}{x}$, and $d_2\,\mathrm{Q}\,\psi(y;z)$.

* For each natural d and s we have: $!\{\{e_2\}(d)\}$, and it is follows from $\langle q;s\rangle \in\overset{\circ}{a}$ and $s \in^0 a$ that $\{\{j_2p\}(d)\}(q)$ is defined and realizes the formula $\varphi(d;y)$ for some $y^1 \in S$. Let us define: $\{\{e_2\}(d)\}(q) = j(q_1;q_2)$ where $q_1 = q$ and $q_2 = \{\{j_2p\}(d)\}(q)$. Then we get:

$$q_2\,\mathrm{Q}\,\varphi(d;y),\ \text{and}\ \langle q_1;d\rangle \in\overset{\circ}{x},\ \text{i.e.}\ j_2\{\{e_2\}(d)\}(q)\,\mathrm{Q}\,\varphi(d;y).$$

So, the conditions (i)-(iv) are satisfied, and $e\,\mathrm{Q}\,Coll$.

$$4.\ \ Sep\ \ \forall x^1\exists y^1[\forall u^0(u\in^0 y\equiv(u\in^0 x)\wedge\varphi(u))\wedge\forall z^1(z\in^1 y\equiv(z\in^1 x)\wedge\psi(z))].$$

Let us find a natural e such that $e\,\mathrm{Q}\,Sep$. Fix an arbitrary x^1, and number h_x such that $\overset{\circ}{x}\ .ext.h_x$. Then it must be: $\{e\}(h_x)$ is defined and realizes the formula:

$$\exists y^1[\forall u^0(u\in^0 y\equiv(u\in^0 x)\wedge\varphi(u))\wedge\forall z^1(z\in^1 y\equiv(z\in^1 x)\wedge\psi(z))].$$

By definition this means that there is some set y such that the following conditions must be satified:

0) $\{e\}(h_x)=j(e^*;e^{**})$;
1) $\overset{\circ}{y}\ .ext.e^{**}$.
2) $e^*\,\mathrm{Q}[\forall u^0(u\in^0 y\equiv(u\in^0 x)\wedge\varphi(u))\wedge\forall z^1(z\in^1 y\equiv(z\in^1 x)\wedge\psi(z))]$.
2.0) $e^*\ =\ j(e_1;e_2)$;
2.1) $e_1\,\mathrm{Q}\,\forall u^0(u\in^0 y\equiv(u\in^0 x)\wedge\varphi(u))$, i.e. for each natural q we have:
2.1.0) $\{e_1\}(q)$ is defined, and $\{e_1\}(q)=j(e_{11};e_{12})$;
2.1.1) $e_{11}\,\mathrm{Q}[q\in^0 y\to(q\in^0 x)\wedge\varphi(q)]$, i.e. for each natural r we have:

$$\text{if } \langle r;q\rangle\in\overset{\circ}{y} \text{ and } q\in y \text{ then } !\{e_{11}\}(q) \text{ and } \{e_{11}\}(q)\,\mathrm{Q}[(q\in x)\wedge\varphi(q)]. \tag{1}$$

Let us try to satisfy the following stronger condition:

$$\text{if } \langle r;q\rangle\in\overset{\circ}{y} \text{ then } !\{e_{11}\}(q) \text{ and } \{e_{11}\}(q)\,\mathrm{Q}[(q\in^0 x)\wedge\varphi(q)\ ]. \tag{2}$$

2.1.2) $e_{12}\,\mathrm{Q}[(q\in x)\wedge\varphi(q)\to q\in^0 y]$, i.e. for arbitrary natural number r:

$$\text{if } s\,\mathrm{Q}[(q\in^0 x)\wedge\varphi(q)] \text{ and } (q\in^0 x)\wedge\varphi(q) \text{ then} \tag{3}$$

$$\{e_{12}\}(s) \text{ is defined and } \{e_{12}\}(s)\,\mathrm{Q}(q\in^0 y).$$

Let us try to satisfy the following stronger condition:

$$\text{if}\quad s\,\mathrm{Q}[(q\in^0 x)\wedge\varphi(q)]\quad\text{then}\quad !\{e_{12}\}(s)\quad\text{and}\quad\{e_{12}\}(s)\,\mathrm{Q}(q\in^0 y). \tag{4}$$

2.2) $e_2\,\mathrm{Q}\,\forall z^1(z\in^1 y\equiv(z\in x)\wedge\psi(z))$, i.e. for each set z, and for each number h_z such that $z.ext.h_z$ we have: $\{e_2\}(h_z)$ is defined and
2.2.0) $\{e_2\}(h_z)\ =\ j(e_{21};e_{22})$;
2.2.1) $e_{21}\,\mathrm{Q}[z\in^1 y\to(z\in^1 x)\wedge\psi(z)]$, i.e. for each natural number r:

$$\text{if } \langle r;\overset{\circ}{z}\rangle\in\overset{\circ}{y}, \text{ and } z\in y \text{ then } !\{e_{21}\}(r)\text{and}\{e_{21}\}(r)\,\mathrm{Q}(z\in x)\wedge\psi(z).$$

Let us try to satisfy the following stronger condition:

$$\text{if } \langle r;\overset{\circ}{z}\rangle\in\overset{\circ}{y}, \text{then } !\{e_{21}\}(r) \text{ and } \{e_{21}\}(r)\,\mathrm{Q}(z\in x)\wedge\psi(z). \tag{5}$$

2.2.2) $e_{22}\,\mathrm{Q}[(z\in^1 x)\wedge\psi(z)\to z\in^1 y]$, i.e. for each natural number s:

$$\text{if } s\,\mathrm{Q}[(z\in^1 x)\wedge\psi(z)] \text{ and } (z\in^1 x)\wedge\psi(z) \text{ then } !\{e_{22}\}(s) \text{ and } \langle s;\overset{\circ}{z}\rangle\in\overset{\circ}{y},$$

Let us try to satisfy the following stronger condition:

$$\text{if } \ s\,\mathrm{Q}(z \in^1 x) \wedge \psi(z) \text{ then } !\{e_{22}\}(s) \text{ and } \langle \{e_{22}\}(s); \overset{\circ}{z} \rangle \in \overset{\circ}{y}\ .$$

Let us define: $\{e_{ij}\}(q) := q \ \ for \ \ i,j = 1,2,3,$ and $y = y_1 \cup y_2,$ where

$$y_1 \coloneqq \{\langle r; \overset{\circ}{z} \rangle | r\,\mathrm{Q}(z \in^1 x) \wedge \psi(z)\},$$

and

$$y_2 \coloneqq \{\langle r; q \rangle | r\,\mathrm{Q}(q \in^0 x) \wedge \varphi(q)\}.$$

Now we have to check only that $y_1 \in \Delta$ and $y_2 \in \Delta$, therefore, $y \in \Delta$, and so, $\overset{\circ}{y} = y$. To do that, we need in the following

Lemma For each formula φ of our language there is a number ϱ such that for all sets z, y, and all $\overrightarrow{h}$ it satisfies:

$$\text{if } \ \overset{\circ}{z} \underset{\gamma}{\overset{\overrightarrow{h}}{\sim}} \overset{\circ}{y} \ \text{ and } \ e\,\mathrm{Q}\,\varphi(z) \ \text{ then } \ \varrho(e;h) \ \text{ is defined, and } \ \varrho(e;\overrightarrow{h}\,)\,\mathrm{Q}\,\varphi(y).$$

The number ϱ recursively depends of parameters $\overrightarrow{n}$, $h_{\overrightarrow{w}}$ and h_a from φ.

Proof. A simple induction on the length of the formula φ. $\square$

So, the conditions (2), (4), (5) and (2.4) are satisfied. $\square$

3. THE ADMISSIBILITY OF CHURCH RULE AND OTHER EFFECTIVITY PROPERTIES.

Theorem 2. *The following effectivity properties hold for $ZFI2^c$:*

1. DP: let $\varphi \vee \psi$ be a formula of the language of the theory $\mathbb{ZFI}2^c$ without any parameters. If $\mathbb{ZFI}2^c \vdash \varphi \vee \psi$ then $\mathbb{ZFI}2^c \vdash \varphi$ or $\mathbb{ZFI}2^c \vdash \psi$.

2. EP: let $\exists x^0 \varphi(x)$ be a formula of the language of the theory $\mathbb{ZFI}2^c$ without any parameters. If $\mathbb{ZFI}2^c \vdash \exists x^0 \varphi(x)$ then there is some number n such that $\mathbb{ZFI}2^c \vdash \varphi(n)$.

3. CR: let φ be a formula of the language of the theory $\mathbb{ZFI}2^c$ without as parameters only x and y of sort 0. If $\mathbb{ZFI}2^c \vdash \forall x^0 \exists y^0 \varphi(x;y)$ then there is some effectively found number e such that $\mathbb{ZFI}2^c \vdash \forall x^0 \varphi(x; \{e\}(x))$.

4. MR: let $\varphi(x)$ be a formula of the language of the theory $\mathbb{ZFI}2^c$ with only one number parameter x. If $\mathbb{ZFI}2^c \vdash \forall x(\varphi(x) \vee \neg\varphi(x))$ and $\mathbb{ZFI}2^c \vdash \neg\neg\exists x \varphi(x)$ then $\mathbb{ZFI}2^c \vdash \exists x \varphi(x)$

Proof. 1. Let $\mathbb{ZFI}2^c \vdash \varphi \vee \psi$, where formula $\varphi \vee \psi$ does not contains any parameters. Then by theorem 1 there is a natural r such that $\mathbb{ZFI}2^c \vdash r\,\mathrm{Q}(\varphi \vee \psi)$. It means by definition that

$$\mathbb{ZFI}2^c \vdash [j_1 e = 0 \to \varphi \wedge j_2 e\,\mathrm{Q}\,\varphi] \wedge [\neg j_1 e = 0 \to \varphi \wedge j_2 e\,\mathrm{Q}\,\psi].$$

If $j_1 e = 0$ then $\mathbb{ZFI}2^c \vdash [\varphi \wedge j_2 e\,\mathrm{Q}\,\varphi]$, and, in particulary, $\mathbb{ZFI}2^c \vdash \varphi$. Analogously, if $\neg j_1 e = 0$ then $\mathbb{ZFI}2^c \vdash \psi$.

2. Let $\mathbb{ZFI}2^c \vdash \exists x^0 \varphi(x)$, where formula $\exists x^0 \varphi(x)$ does not contains any parameters. Then by theorem 1 there is a natural r such that

$$\mathbb{ZFI}2^c \vdash r\,\mathrm{Q}\,\exists x^0 \varphi(x).$$

It means by definition that

$$\mathbb{ZFI}2^c \vdash j_2 r\,\mathrm{Q}\,\varphi(j_1(r)) \wedge \varphi(j_1(r)).$$

So, we have:

$$\mathbb{ZFI}2^c \vdash \varphi(j_1(r)).$$

3. Let $\mathbb{ZFI}2^c \vdash \forall x^0 \exists y^0 \varphi(x;y)$, where formula $\forall x^0 \exists\, y^0 \varphi(x;y)$ does not contains any parameters. Then by theorem 1 there is a natural r such that

$$\mathbb{ZFI}2^c \vdash r\,\mathrm{Q}\,\forall x^0 \exists y^0 \varphi(x;y).$$

It means by definition that

$$\mathbb{ZFI}2^c \vdash \forall d^0 [!\{r\}(d) \wedge j_2(\{r\}(d))\,\mathrm{Q}\,\varphi(d; j_1(\{r\}(d))) \wedge \varphi(d; j_1(\{r\}(d)))].$$

In particular, we have:

$$\mathbb{ZFI}2^c \vdash (\forall d \in \omega)[!\{r\}(d) \wedge \varphi(d; j_1(\{r\}(d)))].$$

Let $\{e\}(d) \leftrightharpoons j_1(\{r\}(d))$. Then $\mathbb{ZFI}2^c \vdash \forall x^0 \varphi(x; \{e\}(x))$.

4. Let $\mathbb{ZFI}2^c \vdash \forall x^0(\varphi \vee \neg\varphi)$ and $\mathbb{ZFI}2^c \vdash \neg\neg\exists x^0 \varphi$, where the formula φ haves, for example, two parameters x and y of sort 0. Then we have: $\mathbb{ZFI}2^c \vdash \forall y \forall x^0(\varphi(x;y) \vee \neg\varphi(x;y))$ and $\mathbb{ZFI}2^c \vdash \forall y^0 \neg\neg\exists x^0 \varphi$. By CR there is a number m such that in $\mathbb{ZFI}2^c$ we have:

$$\vdash \forall y \forall x^0 (!\{m\}(x;y) \wedge [(\{m\}(x;y) = 0 \to \varphi(x;y)) \wedge (\neg(\{m\}(x;y) = 0) \to \varphi(x;y)).$$

So,

$$\mathbb{ZFI}2^c \vdash \forall y^0 \forall x^0 (!\{m\}(x;y) \wedge [(\{m\}(x;y) = 0 \equiv \varphi(x;y)).$$

But, by our hypothesis, also $\mathbb{ZFI}2^c \vdash \forall y^0 \neg\neg\exists x^0 \varphi(x;y)$, therefore, we have:

$$\mathbb{ZFI}2^c \vdash \forall y^0 \neg\neg\exists x^0\ [\{m\}(x;y) = 0].$$

By weak Markov rule (WMR) without parameters for $ZFI2^c$ we get:

$$\mathbb{ZFI}2^c \vdash \forall y^0 \exists x^0 \{m\}(x;y) = 0,$$

and therefore, we get:

$$\mathbb{ZFI}2^c \vdash \forall y^0 \exists x^0 \varphi(x;y). \Box$$

Corollary 1. . *The effectivity properties with parameters for* $\mathbb{ZFI}2$.

1. DP: let $\varphi \vee \psi$ be a formula of the language of the theory $\mathbb{ZFI}2$ may be with set parameters. If $\mathbb{ZFI}2^c \vdash \varphi \vee \psi$ then $\mathbb{ZFI}2^c \vdash \varphi$ or $\mathbb{ZFI}2^c \vdash \psi$.

2. EP: let $\exists x^0 \varphi(x)$ be a formula of the language of the theory $\mathbb{ZFI}2$ may be with set parameters. If $\mathbb{ZFI}2 \vdash \exists x^0 \varphi(x)$ then there is some number n such that $\mathbb{ZFI}2 \vdash \varphi(n)$.

3. CR: let φ be a formula of the language of the theory $\mathbb{ZFI}2$ without as number parameters only x and y. If $\mathbb{ZFI}2 \vdash \forall x^0 \exists y^0 \varphi(x;y)$ then there is some effectively found number e such that $\mathbb{ZFI}2 \vdash \forall x^0 \varphi(x; \{e\}(x))$.

4. MR: let $\varphi(x)$ be a formula of the language of the theory $\mathbb{ZFI}2$ with only one number parameter x, and may be some parameters of sort 1. Then we have: if $\mathbb{ZFI}2 \vdash \forall x(\varphi(x) \vee \neg\varphi(x))$ and $\mathbb{ZFI}2 \vdash \neg\neg\exists x \varphi(x)$ then $\mathbb{ZFI}2 \vdash \exists x \varphi(x)$.

Proof. As in subsection 2.1 we consider the formula φ' that is created from φ by replacement of all parameters in φ by different free constants. $\Box$

So, we have proved the following

3.1. Main theorem

Theorem 3. *1. DP: let $\varphi \vee \psi$ be a formula of the language of the theory $\mathbb{ZFI}2$ may be with set parameters. If $\mathbb{ZFI}2 \vdash \varphi \vee \psi$ then for each derivation of this formula there is some effectively found derivation of φ or some effectively found derivation of ψ.*

2. EP: let $\exists x^0 \varphi(x)$ be a formula of the language of the theory $\mathbb{ZFI}2$ may be with set parameters. If $\mathbb{ZFI}2 \vdash \exists x^0 \varphi(x)$ then for each derivation there are some effectively found number n and derivation $\mathbb{ZFI}2$ of $\varphi(n)$.

3. CR: let φ be a formula of the language of the theory $\mathbb{ZFI}2$ without as number parameters only variables x and y . If $\mathbb{ZFI}2 \vdash \forall x^0 \exists y^0 \varphi(x; y)$ then for each derivation of this formula there are some effectively found number e and derivation in $\mathbb{ZFI}2$ of the formula $\forall x^0 \varphi(x; \{e\}(x))$.

4. MR: let $\varphi(x)$ be a formula of the language of the theory $\mathbb{ZFI}2$ with only one number parameter x, and may be some parameters of sort 1. Then we have: if $\mathbb{ZFI}2 \vdash \forall x(\varphi(x) \vee \neg\varphi(x))$ and $\mathbb{ZFI}2 \vdash \neg\neg\exists x \varphi(x)$ then $\mathbb{ZFI}2 \vdash \exists x \varphi(x)$, where derivation of consequence of MR can be found effectively by derivations of it's antecedent.

Let us recall **dc** is double complement principle and $\mathbb{M}^-$ is weak Markov principle [4]. Let us note that this theorem is generalized on a number of other theories in the language of $\mathbb{ZFI}2$. For example, it is true for $\mathbb{ZFI}2 + \mathbf{dc}$ as well $\mathbb{ZFI}2 + \mathbb{M}^-$. This will be shown in [7].

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