Информационные процессы, Том 9, № 3, 2009, стр. 216–223.

THE M/D/1 PROCESSOER SHARING QUEUE REVISITED [1]

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Received September 22, 2006

Abstract—Starting from previous results of studying the M/G/1 queue with egalitarian processor sharing (EPS), we consider a special case: the M/D/1—EPS, and show how to obtain the (conditional) sojourn time distribution for this special case from more general results. Some new properties of such queues are discovered. We also establish some connections between the M/D/1—EPS queue and some known results from geometrical probability and uniform spacings.

1. INTRODUCTION

One of the most interesting service disciplines in queueing theory is that of egalitarian processor sharing (EPS): when $n > 0$ jobs are present in the system, then every job is being served with rate $1/n$. In other words, all these jobs simultaneously receive $1/n$ times the rate of service which a solitary job in the processor (server) would receive. Jumps of the service rate occur at the instants of arrivals and departures from the system. Therefore, the rate of service received by a specific job fluctuates with time and, importantly, its sojourn time depends not only on the jobs in the server at its time of arrival there, but also on subsequent arrivals shorter of which can overtake a specific job. This makes the EPS queue intrinsically harder to analyse than, say, the classical First Come—First Served (FCFS) queue or many other classical disciplines. The system works in steady state.

EPS queue was introduced by Kleinrock [1] in 1964 and has been the subject of much research over the past 40+ years. In this model one of the main measures of performance is a (tagged) job's sojourn time distribution, conditioned on that job's service time (job's size). The (stationary) sojourn time is the time the tagged job leaves the system after being served, assuming the job arrives at time zero.

We denote by $V(u)$ the conditional sojourn time, with u being the service time. If the tagged job arrived to an empty system and no further arrivals occurred in the time interval $[0, u]$, then $V(u) = u$. But in general $V(u) > u$ as the tagged job must share the capacity of the server. We denote by $\beta(u)$ the service time density, by $v(x|u)$ the conditional sojourn time density, and by $v(x) = \int_0^x v(x|u)\beta(u)\,du$ the unconditional sojourn time density. In general, $v(x|u)$ has a probability mass along $x = u$, but $v(x)$ is generally continuous function.

The M/M/1—EPS queue assumes Poisson arrivals and i.i.d. service times with density $\beta(u) = \mu e^{-\mu u}$. An expression for $\mathbb{E}[e^{-sV(u)}]$ (that is, for the Laplace transform (LT) of $v(x|u)$) is known since 1970 (see, e.g., Kleinrock's book [2, Eq. (4.19)] (1976)).

[1] This paper is an extended version of the talk presented on Dobrushin International Conference held in Moscow at July 2009.

A more difficult model is the M/G/1—EPS queue, where the service density is general. This was independently analysed by the author in [3] (1981), [4] (1983) and by Schassberger [5] (1984) by means of completely different new analytical methods (in particular, the papers [3, 4] use the view of the EPS queue as a branching process.) These authors obtained an explicit, albeit complicated, expression for $\mathbb{E}[\mathrm{e}^{-sV(u)}]$. Inverting the LT leads to an expression for $v(x|u)$ as a contour integral (see Theorem 2.1), but the integrand is a nonlinear function of another contour integral, which is in turn defined in terms of the LT of the service density.

In this paper we will give a brief derivation of the Laplace–Stieltjes transform (LST) of the conditional sojourn time distribution in the M/D/1—EPS queue with deterministic service density. This was derived by the author (see [6, p. 73]) and more recently in [7, 8]. However, in most cases authors use arguments that are specific to the case $G = D$. But these results also follow easily from the general M/G/1—EPS model. This paper shows how such special results can be obtained from the general results. We also give some insight to the properties of the main ingredient of sojourn time in the M/D/1—EPS queue, which are related to well–known problems from geometrical probability and uniform spacings.

2. PRELIMINARIES

In this section we give a short representation about the main results of the determination of the stationary sojourn time distribution (in terms of double Laplace transforms) for the M/G/1—EPS queue.

Let jobs arrive to the single server according to a Poisson process $N = \{N(t) : t \geq 0\}$ with the rate $\lambda > 0$. Their sizes are i.i.d. random variables with a general distribution function $B(u) = \mathbb{P}(B \leq u)$, $(B(0) = 0,\ B(\infty) = 1)$ with the mean $\beta_1 < \infty$ and the LST $\beta(s) \stackrel{\triangle}{=} \int_{0-}^{+\infty} \mathrm{e}^{-su}\, dB(u)$.[2] We assume that $\rho = \lambda\beta_1 < 1$.

We recall that $V(u)$ denotes the conditional sojourn time of a job of the size u upon its arrival. The LST of $V(u)$ is defined by $v(s, u) \stackrel{\triangle}{=} \mathbb{E}[\mathrm{e}^{-sV(u)}]$ for Re $s \geq 0$ and $u \geq 0$.

Let $\pi(s)$ be the LST of the busy period distribution. In other words, it is the positive root of the well–known Kendall–Takács functional equation

$$\pi(s) = \beta(s + \lambda - \lambda\pi(s)) \tag{2.1}$$

with the smallest absolutely value.

To obtain the LST $v(s, u) \stackrel{\triangle}{=} \int_0^\infty \mathrm{e}^{-sx}\, d\mathbb{P}(V(u) \leq x)$ the following (non–trivial) decomposition of the random variable $V(u)$ was carried out. We tagged some (virtual) job of the length u and examined the process of the accumulation of its attained service time. We assume that this tagged job enters into the EPS system at time $t = 0$ under the condition that it meets at its arrival time $n \geq 0$ other jobs (the ancestors) in the system with the remaining sizes which lie in infinitesimal neighbourhood of the points $x_1, \ldots, x_n$ (that is, the EPS system is in the state $(n; x_1, \ldots, x_n)$ if $n > 0$ or the system is empty if $n = 0$). Then the sojourn time of the tagged virtual job is decomposed as:

$$V_n(u|(n; x_1, \ldots, x_n)) \stackrel{d}{=} \sum_{i=1}^{n} \Phi(x_i, u) + D(u). \tag{2.2}$$

[2] We assumed that $B(\cdot)$ has no atom in the origin. For otherwise, the pattern of busy and idle periods is essentially the same as in a queueing process for which arrival rate is reduced to $\lambda[1 - \mathbb{P}(B = 0)]$, and service time has the distribution of B given that $B > 0$.

Here $\Phi(x,u)$ is the sum of increments of attained service time (an age) of a job–ancestor of the initial size x and its direct jobs–descendants for the time interval during which the remaining size of other ancestor (say, the tagged job) is reduced by u. This random variable may be also considered as some (Markovian) functional of the corresponding branching process which describes its total lifetime. However, it is more simple to interpret the random variable $\Phi(x,u)$ as a duration of some *terminating* (sub)busy period initiated by ancestor with the size x. It is terminated at time when the increment of the attained service time of the tagged job reaches the level u. The probabilistic structure of the ingredients of such a busy period is reminiscent of the structure of the components of a standard busy period, but with the important difference that each subsequent component depends on the termination time of a branching process and the size of a descendant. Therefore a subsequent component is *stochastically smaller* than a preceding component (in the sense of some stochastic order relation of type $\overset{1}{\leq}$ for the distribution functions.)

As $u \to \infty$, then the random variable $\Phi(x,u)$ is reduced to the standard busy period with the fixed size x of the job which opens it. The random variable $\Phi(x,u)$ does not depend on x as $x \geq u$. For convenience, the special notation for this case was introduced in Eq. (2.2):

$$D(u) \overset{d}{=} \Phi(x,u) \quad \text{for} \quad x \geq u. \tag{2.3}$$

The components of the stochastic equality (2.2) (which were called *delay elements* in [4]) are independent of each other. The independence of these random variables is an another non–trivial fact which was elegantly proved by means of two ways: using an equiprobable random selection mechanism for a distinction of jobs–descendants [4, pp.138–139], and using the *random time change* [6, §2.8].

To find the distributions of the components of the decomposition (2.2), we need to derive and solve some system of differential equations (with initial–boundary conditions). Let $\varphi(s,x,u) \overset{\Delta}{=} \mathbb{E}[\mathrm{e}^{-s\Phi(x,u)}]$ and $\delta(s,u) \overset{\Delta}{=} \mathbb{E}[\mathrm{e}^{-sD(u)}]$. Then

$$\frac{\partial\varphi(s,x,u)}{\partial x} + \frac{\partial\varphi(s,x,u)}{\partial u} + \left[s + \lambda - \lambda\int_0^\infty \varphi(s,y,u)\,dB(y)\right]\varphi(s,x,u) = 0, \tag{2.4}$$

$$\frac{\partial\delta(s,u)}{\partial u} + \left[s + \lambda - \lambda\int_0^\infty \varphi(s,y,u)\,dB(y)\right]\delta(s,u) = 0, \tag{2.5}$$

$$\delta(s,0) = \varphi(s,0,u) = \varphi(s,x,0) = 1. \tag{2.6}$$

Thus

$$\mathbb{E}[\mathrm{e}^{-sV(u)}|(n;x_1,\ldots,x_n)] = \delta(s,u)\prod_{i=1}^{n}\varphi(s,x_i,u), \quad \operatorname{Re} s > 0. \tag{2.7}$$

¿From here we obtain after removing the condition on $(n;x_1,\ldots,x_n)$ (that is, after averaging on the stationary distribution density of the Markov process of the number of jobs with the remaining sizes which lie in infinitesimal neighbourhood of the points $x_1,\ldots,x_n$) the following statement.

Theorem 2.1. *When $\rho < 1$, then*

$$v(s,u) \overset{\Delta}{=} \mathbb{E}[\mathrm{e}^{-sV(u)}] = (1-\rho)\delta(s,u)\left[1 - \rho\int_0^\infty \varphi(s,x,u)\frac{(1-B(x))}{\beta_1}dx\right]^{-1}, \tag{2.8}$$

where

$$\varphi(s,x,u) = \begin{cases} \delta(s,u) & \text{for } x \geq u, \\ \delta(s,u)/\delta(s,u-x) & \text{for } x < u, \end{cases} \tag{2.9}$$

and

$$\delta(s,u) = \mathrm{e}^{-u(s+\lambda)}/\psi(s,u), \quad u \geq 0 \tag{2.10}$$

are the solutions of the system of equations (2.4) *and* (2.5) *(together with* (2.6)*). Here* $\psi(s,u)$ *is the LST (with respect to* x*) of some function* $\Psi(x,u)$ *of two variables (possessing the probability density on variable* x*), which, in turn, has a LT with respect to* u *(argument* q*)*

$$\tilde{\psi}(s,q) = \frac{q+s+\lambda\beta(q+s+\lambda)}{(q+s+\lambda)(q+\lambda\beta(q+s+\lambda))} \quad (s \geq 0,\ q > -\lambda\pi(s)). \tag{2.11}$$

Eq. (2.8) is a representation of the random variable $V(u)$ in the form of some geometric random sum. Here we do not consider various subtleties of the proof (all this was described in the works cited). It is worth noting that the function $\tilde{\psi}(s,q)$ in (2.11) is given in the form of the two–dimensional transform of a function $\Psi(x,u)$)

$$\tilde{\psi}(s,q) = \int_0^\infty \int_0^\infty \mathrm{e}^{-sx-qu} d_x\Psi(x,u)du. \tag{2.12}$$

In other words, $\psi(s,u)$ is the Laplace transform inversion operator, namely, $\psi(s,u) = \mathcal{L}^{-1}(\tilde{\psi}(s,q))(s,u)$, that is, the contour Bromvich integral

$$\psi(s,u) = \frac{1}{2\pi i}\int_{-i\infty+0}^{+i\infty+0} \tilde{\psi}(s,q)\mathrm{e}^{qu}\, dq.$$

Remark 2.1. Briefly, we have derived the expression for $\mathbb{E}[\mathrm{e}^{-sV(u)}]$ by writing the sojourn time as some functional on a branching process (like the processes by Crump–Mode–Jagers, see, for example [9]).Using the structure of the branching process, we found and solved a system of partial differential equations (of the first order) determining the components of a decomposition of $V(u)$. It leads to $\mathbb{E}[\mathrm{e}^{-sV(u)}]$. Many important details can be found in [6,10] where the stationary solutions are further extended to the time–dependent cases.

In some cases, it can be useful the equivalent forms of (2.9). For example,

$$\varphi(s,x,u) = \mathrm{e}^{-(x\wedge u)(s+\lambda)+\lambda\int_0^{x\wedge u}\varphi_B(s,u-y)\,dy},\ x \in [0,\infty), \tag{2.13}$$

where

$$\varphi_B(s,t) \overset{\triangle}{=} \int_0^\infty \varphi(s,x,t)\,dB(x) = \int_0^t \mathrm{e}^{-\int_{t-x}^t (s+\lambda-\lambda\varphi_B(s,y))\,dy}\,dB(x) + (1-B(t))\mathrm{e}^{-\int_0^t(s+\lambda-\lambda\varphi_B(s,y))\,dy}. \tag{2.14}$$

The equality (2.14) represents the functional equation that must be satisfied by the function $\varphi_B(s,\cdot)$. The function $\varphi_B(s,t)$ is the LST of the distribution of some non–trivial terminating busy period (it terminates at time t) for the M/G/1—EPS queue. The solution of the equation (2.14) was obtained in terms of the function ψ $\quad(\psi(s,t) \overset{\triangle}{=} \exp(-\lambda\int_0^t\varphi_B(s,y)\,dy))$ (more precisely, in terms of the LT for this function, see (2.11)). This also shows that the study of the sojourn time in the M/G/1 queue requires deeper analysis in comparison with an analysis that is expected at first sight.

Remark 2.2. It is worth mentioning that the random variable $D(u)$ in (2.2) constitutes a "main" ingredient of the sojourn time: it has the distribution of the sojourn time of a job with the size u that enters into an empty system. When the system is not empty, the ith job (among the jobs which are sharing the capacity of the processor), having remaining length x_i, "adds" a delay $\Phi(x_i,u) = \Phi(x_i \wedge u, u)$ to the new job's sojourn times.

Next we consider a special case of the M/G/1—EPS queue in equilibrium: the M/D/1 system with egalitarian processor sharing.

3. THE M/D/1—EPS QUEUE

Let us begin from the form of deterministic distribution

$$B(x) = \begin{cases} 0, & 0 \le x < u, \\ 1, & x \ge u. \end{cases}$$

Hence the LST of this distribution has the form $\beta(s) = \exp(-su)$ with the moments $\beta_i = u^i$, $i = 1, 2, \ldots$. The offered load is equal to $\rho = \lambda u < 1$. In this special case, the distributions of conditioned and unconditioned sojourn times coincide, hence we may use $V = V(u)$ to denote the steady–state sojourn time of a job in the queue M/D/1—EPS.

Corollary 3.1. *The LST of the stationary distribution of $V(u)$ in the special case M/D/1—EPS has simpler form in comparison with* (2.8)*.*

$$v(s) = v(s, u) = \frac{(1-\rho)(s+\lambda)^2 \mathrm{e}^{-u(s+\lambda)}}{s^2 + \lambda[s + (s+\lambda)(1-\rho)]\mathrm{e}^{-u(s+\lambda)}}. \tag{3.1}$$

In the case considered, the formula (2.10) takes the form

$$\delta(s, u) = \frac{s+\lambda}{\lambda + s\mathrm{e}^{u(s+\lambda)}}. \tag{3.2}$$

Proof. The solution for $v(s, u)$ for the case M/D/1—EPS can be found from Theorem 2.1 in explicit form. In our case, the equation (2.8) is reduced to the form

$$v(s) = v(s, u) = \frac{(1-\rho)\delta(s, u)}{1 - \lambda\delta(s, u)\int_0^u \frac{dx}{\delta(s, u-x)}} \tag{3.3}$$

where $\delta(s, u)$ is given by (3.2). To obtain (3.2), it is easier to use the equation (3.15) from [4] for the unknown function $\delta(s, u)$ (reflected as (2.29) in [6] or (2.20) in [10]) instead of inverting the function $\tilde{\psi}(s, q)$ that is given by (2.11). (However, such inversion is also possible, see [12, pp. 42–43]. Similar inversion for the case M/M/1—EPS was also executed in [6, p. 74]). We recall Eq. (2.20) from [10] that must be satisfied by an unknown function δ

$$\frac{\partial\delta(s, u)}{\partial u} + \left[s + \lambda - \lambda\int_0^u \frac{\delta(s, u)}{\delta(s, u-y)}\, dB(y) - \lambda\int_u^\infty dB(y)\right]\delta(s, u) = 0. \tag{3.4}$$

Since every solution of Eq. (3.4) must satisfy the estimation

$$\delta(s, u) > \mathrm{e}^{-u(s+\lambda-\lambda\pi(s))} \quad \text{for } \mathsf{Re}\ s > 0, \tag{3.5}$$

then δ can be represented as (2.10) where an unknown function $\psi(s, u) < \mathrm{e}^{-\lambda\pi(s)u}$ for $\mathsf{Re}\ s > 0$. Substituting (2.10) into (3.4) yields the equation that must be satisfied by an unknown function $\psi(s, u)$.

$$\frac{\partial\psi(s, u)}{\partial u} + \lambda\int_0^u \mathrm{e}^{-y(s+\lambda)}\,\psi(s, u-y)dB(y) + \lambda(1 - B(u))\,\mathrm{e}^{-u(s+\lambda)} = 0 \tag{3.6}$$

with the additional conditions $\psi(s, 0) = 1$ and $\psi(0, u) = \mathrm{e}^{-\lambda u}$. These conditions are found from equalities (2.10) and (2.6).

Equation (3.6) is solved by means of the Laplace transform. In our case (M/D/1) equation (3.4) reduces to the form

$$\frac{\partial\delta(s, x)}{\partial x} + (s+\lambda)\delta(s, x) - \lambda\delta(s, x)^2 = 0 \tag{3.7}$$

with the additional condition $\delta(s, 0) = 1$. This is a Bernoulli equation. It is reduced to linear one after division of each term by $\delta(s, x)^2$ and the change of variable $1/\delta(s, x) = u$. The solution of (3.7) is given by (3.2). The final result (3.1) follows after the substitution (3.2) into (3.3). □

Remark 3.1. The general expression for the variance of $V(u)$ in the M/G/1—EPS queue (see Eq. (3.20) in [4] or (2.33) in [10]) was obtained in the form

$$\mathsf{Var}[V(u)] = 2(1-\rho)^{-2} \int_0^u (u-x)(1-W(x))\,dx, \tag{3.8}$$

where

$$W(x) = (1-\rho)\sum_{n=0}^{\infty} \rho^n F^{n*}(x), \tag{3.9}$$

$$F^{n*}(x) = \int_0^x F^{(n-1)*}(x-y)\,dF(y), \tag{3.10}$$

$$F(x) = F^{1*}(x) = \int_0^x f(y)\,dy = \beta_1^{-1}\int_0^x (1-B(y))\,dy.^3$$

Here $F^{0*}(x) = \mathbf{1}(x)$ is the Heaviside's function. We note that (3.9) is the waiting time distribution in classical M/G/1–FCFS queue. The expression (3.8) reduces for the M/D/1—EPS system to the form

$$\mathrm{Var}[V(u)] = \frac{u^2}{(1-\rho)^2} - \frac{2u^2(\mathrm{e}^{\rho}-1-\rho)}{\rho^2(1-\rho)}.$$

Another way of obtaining $\mathrm{Var}[V(u)]$ in the M/D/1—EPS queue is described in [8]. That approach is also based on the results of [4].

Remark 3.2. In addition to [4], we can give the two new interpretation of a random variable $D(u)$ whose LST is given by $\delta(s,u)$ (for the case M/D/1—EPS queue). (See [3], [4] concerning previous interpretation via lifetime of some branching process.) First, it is the sojourn time of the first job that arrives to the empty M/D/1—EPS queue. The explanation of this fact is as follows: until the service requirement of the first job is completed, a number of other jobs may arrive but none leave the system before that time, since under EPS discipline with deterministic service time jobs depart from the system without overtaking, that is, in order of their arrival. Second, the random variable $D(u)$ may be interpreted in terms of the maximal length of the pieces of a stick with length u broken randomly (see, for example, the books [13, 14] for details).

Let $L(t)$ be the number of jobs at time t. Then $D^*(t) = N(t) - L(t)$ be the number of departures by time t ($N(t)$ was introduced in the begin of §2), and $D_1^* = \inf\{t:\ D^*(t) = 1\}$ be the time until the first departure from the EPS queue. The following theorems comment Remark 3.2.

Theorem 3.1. *Let the M/G/1—EPS queue starts from the state $L(0) = 1$. Then the distribution of D_1^* is given by the LST (3.2) for $u = t$ in the case $G = D$, and by*

$$\mathbb{E}[\mathrm{e}^{-sD_1^*}] = \frac{(s+\lambda)\beta(s+\lambda)}{s+\lambda\beta(s+\lambda)} \tag{3.11}$$

in general case.

Proof. Omitted. (We mean the proof of (3.11) because the assertion for $G = D$ follows from Remark 3.2 and definition of $D(u)$ in (2.3).) □

[3] The d.f. $F(x)$ has many different names. For instance, it is called the random modification of the distribution $B(x)$ or excess or the integrated tail of $B(x)$ or the forward recurrence time, etc.

Consider a stick of length u that is randomly broken into n pieces with lengths $S_1, S_2, \ldots$. It is known that the distribution of the largest piece (maximal uniform spacing) $\mathbb{P}(\max_{i=1,\ldots,n} S_i \le x)$ is given by Whitworth's formula (1901)[4] [13, p. 31], [14, p. 29]:

$$\mathbb{P}(\max_{i=1,\ldots,n} S_i \le x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(1 - k\frac{x}{u}\right)_+^{n-1}, \tag{3.12}$$

where $x_+ = \max(0, x)$.

It holds

Theorem 3.2. *For the M/D/1—EPS queue as $t \ge 0$,*

$$\mathbb{P}(D(u) \le t) = 1 - \mathrm{e}^{-\lambda t} \mathbf{1}_{\{t \le u\}} + \mathrm{e}^{-\lambda t} \sum_{n=1}^{\infty} \sum_{k=0}^{n+1} \frac{(\lambda t)^n}{n!} \binom{n+1}{k} (-1)^k \left(1 - k\frac{x}{t}\right)_+^n. \tag{3.13}$$

Main points of Proof. Consider a stick of length u that is randomly broken into n pieces with lengths $S_1, S_2, \ldots$. The breaking points are given by a sample of size $n-1$ from a uniform distribution on $[0, t]$. Let $E, E_1, E_2, \ldots$ denote iid random variables (exponentially distributed) with mean $1/\lambda$. Let also $E_{(1)}, \ldots, E_{(n)}$ denote the order statistics such that $E_{(k)}$ is the smallest value among $E_1, \ldots E_n$. Besides, let $\bar{S}_n = \sum_{k=1}^{n} E_k$.

It is known

Proposition 3.1. [16]

$$(E_1, E_2, \ldots E_n) \stackrel{d}{=} (X_n, X_{n-1} \ldots, X_1), \tag{3.14}$$

where $X_k = (n-k+1)(E_{k-2} - E_{k-1})$ for $1 \le k, n$ and $X_n = nE_{(1)}$.

It follows from this Proposition

Corollary 3.2.

$$\max_{i=1,\ldots,n} S_i \stackrel{d}{=} \sum_{i=1}^{n} \frac{S_i}{i}. \tag{3.15}$$

Proof. It is known the following property of uniform spacings [15]

$$(S_1, \ldots, S_n) \stackrel{d}{=} (E_1/\bar{S}_n, \ldots, E_n/\bar{S}_n). \tag{3.16}$$

Therefore we have from Proposition 3.1

$$\sum_{i=1}^{n} \frac{S_i}{i} \stackrel{d}{=} \sum_{i=1}^{n} \frac{E_i/\bar{S}_n}{i} = \frac{\sum_{i=1}^{n} \frac{E_i/}{i}}{\bar{S}_n} \stackrel{d}{=} \frac{\sum_{i=1}^{n} X_{n-i+1}/i}{\bar{S}_n} \stackrel{d}{=} \frac{E_{(n)}}{\bar{S}_n} \stackrel{d}{=} \max_{i=1,\ldots,n} S_i.$$

□

Next we can obtain after some manipulation with Corollary 3.2 and (3.2) from Corollary 3.1 the statement of Theorem 3.2. (Other details are omitted). □

[4] Another interpretation of Whitworth's formula is connected with the determination of a probability that the circle is completely covered by the arcs of lengths x that are attached to each random point on a circle of length u (these n points are randomly located on a circumference). It is the so-called Steven's formula [13] in such interpretation.

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