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Second Order Optimal Sequential Model Choice and Change-point Detection [1]

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Abstract—An asymptotic lower bound is proved involving second additive term of the order $\sqrt{|\ln \alpha|}$ for the mean length of a controlled sequential strategy s for discrimination between two statistical models in a very general nonparametric setting. Small parameter α is the maximal error probability of s. A sequential strategy is constructed attaining (or almost attaining) this asymptotic bound uniformly over the distributions of models including those from the indifference zone. These results are extended for general loss function $g(N)$ of the length N of strategies growing not faster at infinity, than some power. Applications of this results to change-point detection and testing homogeneity is outlined.

1. INTRODUCTION

1.1. Setting of the Problem

There are numerous references on sequential hypotheses testing and quick detection of parameter changes in stochastic system. The aim of the present paper is three-fold:

1. to construct second-order optimal sequential strategies strengthening the traditional ones;
2. to do this for a non-parametric setting with control, indifference zone and general risk;
3. to show applicability of the above ideas and constructions for change-point detection and testing homogeneity.

We begin with a brief review of sequential discrimination between a **finite** set of distributions which illuminates further general exposition.

Let us outline the setting of the problem following [1] where first order optimal tests were constructed.

No control case. Let $(X, \mathcal{B}, \mu)$, $X \subset \mathbf{R}$, be a probability space, $(\mathcal{P}, d(\cdot))$ be a metric space, where $\mathcal{P}$ is a subset of the set $\mathcal{A}$ of probability measures absolutely continuous mutually and with respect to μ. Their densities are denoted by the corresponding small letters.

Denote by $\mathbf{E}_P f(X)$ the expectation of $f(X), X$ is a random variable (RV) with distribution P. Let $I(P, Q) = \mathbf{E}_P \log \frac{p(X)}{q(X)}$ be the relative entropy (Kullback–Leibler divergence) with usual conventions (logarithms are to the base e, $0 \log 0 = 0$ etc), and the metric d be *I-uniformly continuous* on $\mathcal{P}$, i.e. for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every pair P, Q from $\mathcal{P}$ if $I(P, Q) < \delta$, $d(P, Q) < \varepsilon$.

The set $\mathcal{P}$ is partitioned into sets $\mathcal{P}_0$, $\mathcal{P}_1$ and the indifference zone $\mathcal{P}_+ = \mathcal{P} \setminus (\mathcal{P}_1 \cup \mathcal{P}_0)$. We test $H_0 : P \in \mathcal{P}_0$ versus $H_1 : P \in \mathcal{P}_1$, any decision is good for $P \in \mathcal{P}_+$.

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Suppose that the distance between the hypotheses is positive, i.e.

$$\inf_{P\in\mathcal{P}_0, Q\in\mathcal{P}_1} d(P,Q) \geq \delta_0 > 0. \tag{1.1}$$

A *strategy* s consists here of the stopping time N and a measurable binary decision δ, $\delta = r, r = 0, 1$, means that H_r is accepted. We assume that observations $X_i, i = 1,\dots,n$ are independent and P-identically distributed (i.i.d.(P)), $P \in \mathcal{P}$, when $N > n$.

For an $\alpha > 0$ introduce α-*strategies* s satisfying

Condition $G(\alpha) : \max_{r=0,1} \sup_{P\in\mathcal{P}_r} \mathbf{P}_P(\delta = 1 - r) \leq \alpha$.

Remark 1. For simplicity of notation we confine ourselves to testing two hypotheses with error probabilities of the same order. A generalized condition $G(\alpha, \mathbf{c}, \mathbf{d}) : \sup_{P\in\mathcal{P}_r} \mathbf{P}_P(\delta = 1 - r) \leq c_r\alpha^{d_r}, c_r > 0, d_r > 0$ can be studied similarly [2]. Generalization to testing several hypotheses see e.g. [3].

Let $\mathbf{E}^s_P N$ be the *mean length* (MEL) of a strategy s. Our first aim is to find an expansion of the lower bound for MEL as $\alpha \to 0$.

Define $I(P,\mathcal{R}) = \inf_{Q\in\mathcal{R}} I(P,Q)$ for $\mathcal{R} \subset \mathcal{P}$; $A(P) = \mathcal{P}_{1-r}$ for $P \in \mathcal{P}_r$ as the alternative set in $\mathcal{P}$ for P. For $P \in \mathcal{P}_+$, if $I(P,\mathcal{P}_0) \leq I(P,\mathcal{P}_1)$, then $A(P) = \mathcal{P}_1$, otherwise, $A(P) = \mathcal{P}_0$. Finally $k(P) = I(P, A(P))$.

We prove in Theorem 1 that for any α-strategy s under mild regularity condition

$$\mathbf{E}^s_P N \geq \frac{|\log\alpha|}{k(P)} + O(\sqrt{|\log\alpha|}). \tag{1.2}$$

In Theorem 3 under stricter regularity conditions we construct the α-strategy s^* attaining equality in (1.2).

Controlled experiments. For controlled testing we suppose that P from $\mathcal{P}$ is a **set of measures** $P = \{P^u, u \in U\}$, $P^u \in \mathcal{A}$, labeled by controls $u \in U = \{1,\dots,m\}$. We use the notation: $p^u(x)$ is the density function of measurements under control u, U^* is the set of mixed controls.

After obtaining the n-th observation experimenter either decides to stop or chooses mixed control for the $(n + 1)$-th experiment. Let $\mathcal{F}_n$ be the σ-algebra generated by the observations and controls up to the time n. We suppose that the $(n + 1)$-th experiment is predictable, i.e. corresponding distribution on U is $\mathcal{F}_n$-measurable, strategy length N is a stopping time under the flow $\mathcal{F}_n$, and a decision δ is $\mathcal{F}_N$-measurable. A strategy s consists now of a rule of mixed control choice $\mathbf{u}(\cdot)$, stopping time N, and decision δ. For more details about constructing a probability space and a controlled strategy see [4].

We assume that metrics d_u are given which are I-uniformly continuous on $\mathcal{P}^u = \{P^u : P \in \mathcal{P}\}$ for each u with respect to the relative entropy and (1.1) holds for the metric $d(P,Q) = \max_{u\in U} d_u(P^u, Q^u)$.

Let $\mathbf{u} = (\kappa_1,\dots,\kappa_m)$, where $\kappa_i \geq 0$ and $\sum_{i=1}^m \kappa_i = 1$, be a mixed control and

$$I_{\mathbf{u}}(P,Q) = \sum_{i=1}^m \kappa_i I(P^i, Q^i).$$

Introduce

$$k^*(P) = \max_{\mathbf{u}\in U^*} I_{\mathbf{u}}(P, A^{\mathbf{u}}(P)) > 0, \tag{1.3}$$

and $u^* = u^*(P)$ as a control such that

$$k^*(P) = I_{u^*}(P, A^{u^*}(P)),$$

where $A^{u^*}(P)$ is the alternative set in $\mathcal{P}^{u^*}$ for P^{u^*}, and (1.1) implies the inequality in(1.3).

Our main result for controlled sequential testing consists in proving a modified lower bound (2.1) and constructing a second order optimal α-strategy s satisfying (2.1).

1.2. Some Results for Finite $\mathcal{P}$

The sequential controlled discrimination between distributions from finite set $\mathcal{P}$ was pioneered in [5]. Subsequent development of first order asymptotically optimal sequential procedure (AOSP1) with the MEL $\mathbf{E}_P^s N$ satisfying $\mathbf{E}_P^s N = |\log(\alpha)|k(P)^{-1}\big(1 + o(1)\big)$ is surveyed [6] in a Bayesian framework. All these results follow from the results of the present paper. It was shown in [8], [7] etc. that the wideness of the class of AOSP1 allows excessive values of the risk function for finite samples. Constructing procedures of the higher (second) order of optimality (AOSP2) became desirable. The results of [9] on Bayesian sequential discrimination between a finite number of distributions (simple hypotheses) imply that (1.4) holds for the optimal strategy with some nonnegative K. Under the additional conditions a strategy is constructed in [10] with MEL exceeding the optimal one in $O(\log|\log(\alpha)|)$.

No-control case. Outlining these results for a finite set $\mathcal{P}$ we begin with a no-control case.

Introduce $z(P,Q,x) = \log\frac{p(x)}{q(x)}$, $L_n(P,Q) = \sum_{i=1}^n z(P,Q,x_i)$. Suppose that $k(P) = I(P,Q_i), i = 1,\dots,l, Q_i \in A(P)$.

Proposition 1. *Let RV $z(P,Q,x)$ possess fourth moments for all P and Q. For any α-strategy s if $l = 1$ then*

$$\mathbf{E}_P^s N \geq \frac{|\log(\alpha)|}{k(P)} + O(1),$$

and if $l > 1$ then

$$\mathbf{E}_P^s N \geq \frac{|\log(\alpha)|}{k(P)} + K\sqrt{\frac{|\log(\alpha)|}{k(P)}}\big(1 + o(1)\big), \tag{1.4}$$

where $K = \mathbf{E}(g(\zeta)) > 0$, $\zeta = (\zeta_1,\dots,\zeta_l)$ is a normally distributed RV with mean $\mathbf{0}$ and covariance matrix $\Sigma = (\Sigma_{ij}), \Sigma_{ij} = \mathbf{E}_P(z(P,Q_i,x)z(P,Q_j,x))$, $g(\zeta) = \max_{i=1,\dots l}\zeta_i$.

Sketch of proof (full proof see in [11]). Define $\mathbf{L}_n(P) = (L_n(P,Q_1),\dots,L_n(P,Q_l))$, $\mathbf{x}_0 = \log(\alpha)\mathbf{1}, \mathbf{1} := (1,\dots,1)$.

Introduce the first time τ, when likelihood ratios of the true distribution P with respect to any alternative exceed the prescribed level $|\log(\alpha)|$. It is not a stopping time (for unknown P) but it is shown in ([9], Lemma 5.1) that any strategy has MEL exceeding $\mathbf{E}_P\tau + const$ for all α.

Therefore the lower bound for MEL is reduced to minimizing the mean time until the first entry into $R^+ = \{\mathbf{x} \in \mathbf{R}^l : x_i \geq 0, i = 1,\dots,l\}$ for the process $\mathbf{x}_0 + \mathbf{L}_n(P)$.

Applying Keener's lower bound ([9], Theorem 2) for our minimization problem we get the results of Proposition 1.

Controlled experiments. Let for simplicity the optimal control u^* be unique and $u_1,\dots,u_t$ be controls in U such that $u^*(P) = (\kappa_1,\dots,\kappa_t,0,\dots,0)$, $\kappa_i > 0$, and $\sum_{i=1}^t \kappa_i = 1$.

Introduce vectors $\mu_i = (I(P^i,Q_1^i),\dots,I(P^i,Q_l^i)), i = 1,\dots,t$, where $Q_j \in A(P)$ be such that $I_{u^*(P)}(P,Q_j) = k(P), j = 1,\dots,l$. Usually it holds $l \geq 2$ for a controlled discrimination.

The **regular case** is defined to be such that $\mu_i, i = 1,\dots,t$, span the subspace L and $\dim(L) = l$. In **degenerate case** $\dim(L) < l$.

An example of degenerate case is the following: $\mathcal{P} = \{P_0,P_1,P_{-1}\}$, $\mathcal{P}_0 = \{P_0\}$, $\mathcal{P}_1 = \{P_1,P_{-1}\}$, $U = \{1,2\}$, and $I(P_0^u,P_r^u) = I, r = 1,-1, u = 1,2$.

Let $F_{st}(P, \mathbf{x})$ be the minimal value of the function

$$f = \sum_{i=1}^{t} \kappa_i \tag{1.5}$$

under the conditions

$$\kappa_i \geq 0, i = 1, \ldots, t, \mathbf{x} + \sum_{i=1}^{t} \kappa_i \mu_i \geq \mathbf{0}, \tag{1.6}$$

and

$$g(\zeta) := F_{st}(P, \mathbf{x}_0 + \zeta) - |\log \alpha| k(P)^{-1}.$$

Proposition 2. *Let RV $z(P^u, Q^u, x)$ possess fourth moments for all P, Q, and u. For any α-strategy s in the regular case*

$$\mathbf{E}_P^s N \geq \frac{|\log(\alpha)|}{k^*(P)} + O(1),$$

while in the degenerate case

$$\mathbf{E}_P^s N \geq \frac{|\log(\alpha)|}{k^*(P)} + K\sqrt{\frac{|\log(\alpha)|}{k(P)}}\big(1 + o(1)\big), \tag{1.7}$$

where $K = \mathbf{E}(g(\zeta)) > 0$, $\zeta = (\zeta_1, \ldots, \zeta_l)$ is a normally distributed RV with mean $\mathbf{0}$ and covariance matrix $\Sigma = (\Sigma_{j_1 j_2}), \Sigma_{j_1 j_2} = \sum_{i=1}^{t} \kappa_i \mathbf{E}_P(z(P^i, Q^i_{j_1}, x) z(P^i, Q^i_{j_2}, x))$.

Sketch of proof (full proof see in ([11])). For finite set U of controls the asymptotically optimal control rule should provide the fastest approach to the positive octant R^+ by the vector composed of likelihood ratios. This problem was studied in [12]. A more simple non-stochastic control problem analogous to (1.5), (1.6) was studied in [9] for sequential discrimination in Bayesian setting.

Let $I_{u^*(P)}(P, Q) > 0$ for all $P, Q \in \mathcal{P}, P \neq Q$. A strategy s^* with MEL $\frac{|\log(\alpha)|}{k(P)} + O(1)$ can be constructed for the regular case under the condition of Proposition 2. We have not enough space to give full description of the strategy and give principal moments only (details see in [11]).

The length of the first stage is $N_1 = A \log(|\log(\alpha)|)$ where A is such that the probability of an error of specifying P is less then $|\log(\alpha)|^{-2}$.

The control rule at the first phase is as follows. This phase has a random number of sub-stages. Every sub-stage begins at the moment of changing the maximum likelihood estimate (ME) $\hat{P}$ based on the previous measurements and the control $u^*(\hat{P})$ is used at this sub-stage. Only the last sub-stage's measurements will be used for the subsequent controls.

Introduce $\mathbf{L}^{(1)} = (L(\hat{P}, Q_1), \ldots, L(\hat{P}, Q_l))$, where $\hat{P}$ is the ML-estimate of P at the end of the first stage, $\{Q_1, \ldots, Q_l\} = A(\hat{P})$, $L(\hat{P}, Q_j) = \sum_{i=\tau}^{N_1} z(\hat{P}^{u_i}, Q_j^{u_i}, x_i)$, τ is a starting time of the last sub-stage and ℓ is a straight line through the points $\mathbf{0}$ and $\mathbf{x}_0 + \mathbf{L}^{(1)}$. On the next two phases we use in general the optimal control from [12] where ℓ is a switching line of the control rule.

Define $\mathbf{L}_n = (L_n(\hat{P}, Q_1), \ldots, L_n(\hat{P}, Q_l))$, where $L_n(\hat{P}, Q_j) = L(\hat{P}, Q_1) + \sum_{i=1}^{n} z(\hat{P}^{u_i}, Q_j^{u_i}, x_i)$.

We stop the phase at the first time when the process $\mathbf{x}_0 + \mathbf{L}_n$ enters R^+ and is sufficiently close to the line ℓ. If $n > 2\left(\min_{P \in \mathcal{P}} k(P)\right)^{-1} |\log(\alpha)|$ then the procedure fails and we return to the starting point.

If the third stage is successful then we stop the measurements. The decision rule consists of accepting the hypothesis containing $\hat{P}$.

In the degenerate case it is necessary to use more complicated control for getting an α-strategy with the property

$$\mathbf{E}_P^s N = \frac{|\log(\alpha)|}{k^*(P)} + K\sqrt{\frac{|\log(\alpha)|}{k^*(P)}} + O(1), \tag{1.8}$$

where K is the same as in (1.7). In contrast to the regular case the optimal control from [12] does not apply for the last phase of the strategy with condition (1.8). A full description of the procedure under the condition (1.8) is given in [11].

A Bayesian controlled sequential discrimination between parametric set $\mathcal{P}$ was studied in [13]. It is proved in [13] that the principal term of the risk function of AOSP1 is the same as for a finite $\mathcal{P}$. Indifference zones were incorporated in [14]. Similar results were obtained for sequential hypotheses testing with an indifference zone in [15] for exponential families $\mathcal{P}$. A generalization onto non-parametric space $\mathcal{P}$ was done in [16] under an assumption that the set $\mathcal{P}$ is convex.

In [1, 17] an AOSP1 was found for non-parametric $\mathcal{P}$ with an indifference zone under general conditions of regularity. We construct AOSP2 in present paper under additional conditions of regularity.

1.3. Brief Outline of Change-Point Detection

Our procedure is also applicable for the non-parametric detection of abrupt change in the distribution of i.i.d. sequence *without* an indifference zone. We use the methodology outlined in [18].

Let the observations $X_1, \ldots, X_n, \ldots$ be independent, and for $n < \nu$ all have a distribution $P_0 \in \mathcal{P}_0$, while all X_n have an unknown distribution $P_1 \in \mathcal{P}_1$ for $n \geq \nu$, where ν is an unknown integer, and (1.1) hold.

Let N be a change-point estimate, and $a^+ = a$, if $a \geq 0$, and $a^+ = 0$ otherwise. Introduce the functional

$$\bar{\mathbf{E}}^s(N) = \sup_{\nu \geq 1} \operatorname{ess\,sup} \mathbf{E}_{P_1}^s \left((N - \nu + 1)^+ \,|X_1, \ldots, X_\nu\right)$$

for $P \in \mathcal{P}_1$ (with index P_1 suppressed) as an optimality criterium of the strategy s under the restriction that for a given $\alpha > 0$

$$\sup_{P \in \mathcal{P}_0} \mathbf{E}_P^s\,(N) \geq \alpha^{-1}. \tag{1.9}$$

Let s be a one-sided sequential strategy for testing the hypothesis $H_1 : P \in \mathcal{P}_1$ versus $H_0 : P \in \mathcal{P}_0$ with a decision δ such that $\sup_{P \in \mathcal{P}_0} \mathbf{P}_P(\delta = 1) \leq \alpha$ and T be its stopping time. For every t we denote by T_t the stopping time of s based on the observations $\mathbf{X}_t = (x_t, x_{t+1}, \ldots)$. Define $N = \inf_t(T_t + t)$ as the stopping time of change-point detection. By Theorem 2 in [18] $\mathbf{E}_P^s\,(N) \geq \alpha^{-1}$ for all $P \in \mathcal{P}_0$. Hence asymptotically optimal strategies for the hypotheses testing are also asymptotically optimal for the change-point detection under the condition (1.9).

Theorem 3 in [18] states under the condition (1.9): $\bar{\mathbf{E}}^s(N) \geq |\log \alpha| I_1^{-1}(1 + o(1))$, where $I_1 = I_1(P_1) = I(P_1, \mathcal{P}_0)$, if the condition C1 in section 2 is satisfied.

In [3] a more extensive survey of connections between the sequential hypotheses testing and the change-point detection is given for finite or parametric $\mathcal{P}$.

We generalize these results for a non-parametric setting and study AOSP2 of change-point detection in subsection 5.2.

2. LOWER BOUND FOR NON-PARAMETRIC HYPOTHESES TESTING

C1. There is $c > 0$ such that $\mathbf{E}_P^u\,(z(P^u, Q^u, X))^2 < c$ for all $P \in \mathcal{P}, Q \in \mathcal{P}$, and $u \in U$.

We prove the following lower bound extending that of [17].

Theorem 1. i. *Under the condition C1 any α-admissible strategy s for the no-control problem satisfies* (1.2) *for every $P \in \mathcal{P}$.*

ii. *For controlled experiments and every $P \in \mathcal{P}$ the following inequality holds*

$$\mathbf{E}_P^s N \geq \frac{|\log \alpha|}{k^*(P)} + O(\sqrt{|\log \alpha|}). \tag{2.1}$$

Proof. i. Choose $Q_n \in A(P)$ such that $I(P, Q_n) \leq k(P) + n^{-1}$.

Two cases are considered separately

A1. $P \in \mathcal{P}_+$,

A2. $P \in \mathcal{P}_r$, $r = 0, 1$.

In the first one we assume for definiteness that $Q_n \in \mathcal{P}_0$. From the definition of $A(P)$ it follows that there exists $Q'_n \in \mathcal{P}_1$ such that $I(P, Q'_n) \leq I(P, Q_n) + n^{-1}$.

Introduce

$$L_k(P, Q_n) = \sum_{i=1}^{k} z(P, Q_n, x_i)$$

and

$$M_0(\alpha) = \begin{cases} \inf\{k : L_k(P, Q_n) \geq -\log \alpha\}, \\ \infty \text{ if } \sup_k L_k(P, Q_n) < -\log \alpha. \end{cases}$$

We define $M_1(\alpha)$ similarly by replacing $L_k(P, Q_n)$ with $L_k(P, Q'_n)$.

It follows from C1 that the upper bound for the overshoot of the level $|\log \alpha|$ (obtained in [12]) does not depend on α. Hence from Wald identity we get

$$\mathbf{E}_P^s M_0(\alpha) \leq \frac{|\log \alpha| + C_1}{I(P, Q_n)}, \tag{2.2}$$

$$\mathbf{E}_P^s M_1(\alpha) \leq \frac{|\log \alpha| + C_1}{I(P, Q'_n)}, \tag{2.3}$$

with the same constant C_1 for all α and n.

Introduce $N_i = \min(M_i(\alpha), N\{\mathcal{D} = 1 - i\})$, where $N\{\mathcal{D} = 1 - i\} = N$ if H_i is rejected and $N\{\mathcal{D} = 1 - i\} = \infty$ if H_i is accepted. It follows from these definitions that

$$M - N \leq \sum_{i=0}^{1} (M_i - N_i), \tag{2.4}$$

where $M = \min(M_0, M_1)$. Similarly to [19] (p. 197) we get

$$\mathbf{E}_P^s N_0 \geq \frac{|\log \mathbf{P}_{Q_n}(N_0 < \infty)|}{I(P, Q_n)}, \quad \mathbf{E}_P^s N_1 \geq \frac{|\log \mathbf{P}_{Q'_n}(N_1 < \infty)|}{I(P, Q'_n)}. \tag{2.5}$$

The error probability of testing simple hypotheses $H'_0 : P = Q_n$ versus $H'_1 : P = Q'_n$ does not exceed α for any α-strategy. Besides, the definition of the stopping times M_i implies that $\mathbf{P}_{Q_n}^s(M_0 < \infty) \leq \alpha$, $\mathbf{P}_{Q'_n}^s(M_1 < \infty) \leq \alpha$. Hence

$$\mathbf{P}_{Q_n}^s(N_0 < \infty) \leq 2\alpha, \mathbf{P}_{Q'_n}^s(N_1 < \infty) \leq 2\alpha \tag{2.6}$$

and we get from (2.2)-(2.6):

$$\mathbf{E}_P^s(M - N) \leq \frac{1 + C_1}{I(P, Q_n)} + \frac{1 + C_1}{I(P, Q'_n)}. \tag{2.7}$$

Thus (2.7) entails:

$$\mathbf{E}_P^s N \geq \mathbf{E}_P^s M - C_2, \tag{2.8}$$

with the same constant C_2 for all α and n.

Introduce $I_n = \max(I(P,Q_n), I(P,Q_n'))$. The definitions of Q_n and Q_n' imply that

$$I_n \leq k(P) + 2n^{-1}. \tag{2.9}$$

It follows from the definitions of the stopping times M_i and the values I_n that

$$|\log\alpha| \leq \max(L_M(P,Q_n), L_M(P,Q_n')) = MI_n +$$

$$\max(M(I(P,Q_n) - I_n) + \zeta_M, M(I(P,Q_n') - I_n) + \zeta_M') \leq MI_n + \max(\zeta_M, \zeta_M'), \tag{2.10}$$

where $\zeta_k = L_k(P,Q_n) - kI(P,Q_n)$ and $\zeta_k' = L_k(P,Q_n') - kI(P,Q_n')$ are martingales. The following inequalities follow for these martingales from condition C1 in Appendix

$$\mathbf{E}_P^s|\zeta_M| \leq C_3(\mathbf{E}_P^s M)^{\frac{1}{2}}, \mathbf{E}_P^s|\zeta_M'| \leq C_3(\mathbf{E}_P^s M)^{\frac{1}{2}}, \tag{2.11}$$

with the same constant C_3 for all α and $\mathbf{E}_P^s M$. Therefore, from (2.10) we get

$$\mathbf{E}_P^s M \geq |\log\alpha| I_n^{-1} - \mathbf{E}_P^s\left(|\zeta_M| + |\zeta_M'|\right) I_n^{-1}.$$

Using the bounds (2.11) we get

$$\mathbf{E}_P^s M \geq |\log\alpha| I_n^{-1} - C_4(\mathbf{E}_P^s M)^{\frac{1}{2}},$$

with the same constant C_4 for all α and n. If we replace $\mathbf{E}_P^s M$ with its upper bound using (2.2) and (2.3), then we get a lower bound

$$\mathbf{E}_P^s M \geq |\log\alpha| I_n^{-1} - C_5(|\log\alpha|)^{\frac{1}{2}}, \tag{2.12}$$

with the same constant C_5 for all α and n. It follows from (2.9), (2.8), and (2.12) that

$$\mathbf{E}_P^s N \geq |\log\alpha| k(P)^{-1} - C_5(|\log\alpha|)^{\frac{1}{2}} - C_2 - 2k(P)^{-2}|\log\alpha| n^{-1}.$$

If a sequence $n = n(\alpha)$ is chosen such that $|\log\alpha|^{\frac{1}{2}} n^{-1} \to 0$ for $\alpha \to 0$, then the theorem follows from the last inequality.

For the case A2 the theorem follows directly from the Wald lower bound for the MEL of the sequential ratio likelihood probability test. The proof of i. is complete.

Proof of ii. We outline only the modifications in the proof i. sufficient for proving ii. similarly to [17]. Let $u^* = u^*(P)$ be a control that provides the maximum in (1.3) and Q_n be a sequence of measures in $A(P)$ such that

$$I^{u^*}(P,Q_n) \leq k^*(P) + n^{-1}.$$

We prove the theorem for $P \in P_+$, other cases are straightforward.

Let $Q_n \in P_0$, $Q_n' \in \mathcal{P}_1$ be such that $I^{u^*}(P,Q_n') < I^{u^*}(P,Q_n)$ as in the proof without control, and $M_i(\alpha)$ be defined as in the previous proof based on the modified statistics

$$L_k(P,Q_n) = \sum_{i=1}^{k} z(P^{u(i)}, Q_n^{u(i)}, x_i),$$

where $u(i)$ is the value of the control for the i-th experiment. We suppose that for $M_i(\alpha) > N$ we use control u^* in experiments after N for definiteness.

If N_i are defined as in the proof of Theorem 1, then (2.4) is valid. Similarly to (2.5) we get

$$\mathbf{E}_P^S \sum_{i=1}^{N_0} z(P^{u(i)}, Q_n^{u(i)}, x_i) \geq |\log \mathbf{P}_{Q_n}(N_0 < \infty)|, \tag{2.13}$$

$$\mathbf{E}_P^S \sum_{i=1}^{N_1} z(P^{u(i)}, Q_n'^{u(i)}, x_i) \geq |\log \mathbf{P}_{Q_n'}(N_1 < \infty)|. \tag{2.14}$$

Define

$$\kappa_l^{(i)} = (\mathbf{E}_P^S N_i)^{-1} \sum_{j=1}^{\infty} \mathbf{P}_P^S(u(j) = l, j \leq N_i)$$

and $\kappa^*(i) = \{\kappa_1^{(i)}, \ldots, \kappa_m^{(i)}\}, i = 0, 1$. From the theorem 2.2.1 in [4] and (2.13), (2.14) we get

$$\mathbf{E}_P^S N_0 I^{\kappa^*(0)}(P, Q_n) \geq |\log \mathbf{P}_{Q_n}(N_0 < \infty)|, \tag{2.15}$$

$$\mathbf{E}_P^S N_1 I^{\kappa^*(1)}(P, Q_n') \geq |\log \mathbf{P}_{Q_n'}(N_1 < \infty)|. \tag{2.16}$$

Formulas (2.2), (2.3) may be rewritten as

$$\mathbf{E}_P^S M_0(\alpha) \leq \frac{|\log \alpha| + C_1}{I^{\kappa'(0)}(P, Q_n)}, \tag{2.17}$$

$$\mathbf{E}_P^S M_1(\alpha) \leq \frac{|\log \alpha| + C_1}{I^{\kappa'(1)}(P, Q_n')}, \tag{2.18}$$

where the control $\kappa'(i)$ is defined as $\kappa^*(i)$, but N_i are replaced with $M_i(\alpha)$. It follows from the definitions that

$$I^{\kappa^*(0)}(P, Q_n) - I^{\kappa'(0)}(P, Q_n) = \frac{\mathbf{E}_P^S(M_0(\alpha) - N_0)}{\mathbf{E}_P^S N_0} \left(I^{\kappa'(0)}(P, Q_n) - I^{u^*}(P, Q_n) \right)$$

and an analogous inequality is valid for $I^{\kappa^*(1)}(P, Q'_n) - I^{\kappa'(1)}(P, Q_n')$. Hence it follows from (2.15)–(2.18) that

$$\mathbf{E}_P^S(M_i(\alpha) - N_i) \leq C_6,$$

with the same constant C_6 for all α and n.

The end of the proof is the same as for the no-control problem.

3. ASYMPTOTICALLY OPTIMAL NON-PARAMETRIC HYPOTHESES TESTING

We use the following regularity conditions:

C2. There exist $t > 0$ and $f > 0$ such that for all $u \in U$ and $P \in \mathcal{P}$ $\mathbf{E}_P \left(\sup_{Q \in \mathcal{P}} \exp(-tz(P, Q, X)) \right) \leq f$.

C3. $z(P, Q, x)$ is differentiable w.r.t. x and

$$D = \int_X z_1(x) \, (a(x)b(x))^{1/2} \, dx < \infty,$$

where

$$z_1(x) = \sup_{Q \in \mathcal{P}} \left| \frac{\partial z(P, Q, x)}{\partial x} \right|, \quad \sup_{P \in \mathcal{P}} \int_{-\infty}^{x} p(t)\mu(dt) \leq a(x), \quad \sup_{P \in \mathcal{P}} \int_{x}^{\infty} p(t)\mu(dt) \leq b(x).$$

C4. There exist the values $b \geq 0$ and $K_1 = K_1(b)$ such that for every n the estimate $\hat{p} = \hat{p}_n$ of the density function of i.i.d.(P) observations $X_1, \ldots, X_n$ with $P \in \mathcal{P}$ can be constructed such that

$$\mathbf{E}_P(I(P, \hat{P})) \leq K_1 n^{-b}. \tag{3.1}$$

Remark 2. If for example the set X is the interval $[0, 1]$ and for $P \in \mathcal{P}$ the function $\log p$ is periodic and belongs to the Sobolev space W_2^r on X, $r \geq 1$, then in [20] verified the condition C4 with $b = \frac{2r}{1+2r}$. Therefore, if additionally to C3 we assume that

$$\int_X \left(\frac{\partial z(P, Q, x)}{\partial x} \right)^2 dx \leq c < \infty$$

with the boundary condition $z(P, Q, 0) = z(P, Q, 1)$ then C4 is valid for $b = \frac{2}{3}$.

If $\mathcal{P}$ determines a smooth quasi-homogeneous family of density functions ([21]), then C4 is valid if we use the methodology of [21] for density's estimation with $b \geq \frac{1}{2}$ depending on smoothness of $\mathcal{P}$.

Remark 3. Usually the estimate $\hat{P}$ is constructed via approximating $\mathcal{P}$ by a parametric exponential family of distributions $\mathcal{A}_m$ of dimension m and using the ML-estimation under the assumption that $P \in \mathcal{A}_m$. Then

$$I(P, \hat{P}) \leq \gamma_1 m^{-r_1} + \gamma_2 \frac{m^{r_2}}{n} \tag{3.2}$$

where γ_1 and γ_2 are numbers, r_1 depends on smoothness of $\mathcal{P}$, and r_2 depends on a choice of a basis for the family $\mathcal{A}_m$. Optimization of (3.2) in m gives (3.1).

Now we introduce our strategy $s^* = s^*(\beta, n)$ depending on the parameters β and n. Procedure s^* consists of conditionally i.i.d. loops. The loop terminating by the event (3.3) is the final loop of s^*. Every loop contains two phases.

Based on the first $L = [\sqrt{|\ln \beta|}] + 1$ observations of a loop we estimate the density function P.

We perform the following test in the second phase. Let us numerate measurements of the second phase anew and introduce $L_k(\hat{P}, Q) = \sum_{i=1}^{k} z(\hat{P}, Q, x_i)$, where $\hat{P}$ is the estimate of P in the first phase. We stop observations at the first moment M such that

$$\inf_{Q \in A_n(\hat{P})} L_M(\hat{P}, Q) > -\log \beta \tag{3.3}$$

or

$$M > 2k(\hat{P})^{-1} |\log \beta| \tag{3.4}$$

and accept the hypothesis H_r (i.e. $\delta = r$) if (3.3) holds and $1 - r$ is the index of the set $A(\hat{P})$. If (3.4) holds then we begin a new loop.

Theorem 2. *For every $P \in \mathcal{P}$ under the conditions C1-C4 and appropriate parameters, s^* is an α-strategy and*

$$\mathbf{E}_P^{s^*} N \leq \frac{|\log \alpha|}{k(P)} + K_2 |\log \alpha|^{1-b/2} + K_3 \sqrt{|\log \alpha|} \tag{3.5}$$

with the same constants K_2 and K_3 for all α.

Proof. Let $\mathcal{F}_k$ be the σ-algebra generated by the first k observations on the second phase and all previous observations, $\mathcal{T}$ be an event that $A(\hat{P}) = A(P)$, and $\mathcal{F}$ be its complement. It follows from (1.1) and well-known Sanov theorem on large deviations for the empirical distribution functions of i.i.d. observations that

$$\mathbf{P}_P^{s^*}(\mathcal{F} | \mathcal{F}_0) \leq C_1 \beta^{b_1}, \tag{3.6}$$

where C_1 and b_1 are positive numbers depending on δ_0 and $\mathcal{P}$ only.

From the definition of $L_k(\hat{P},Q)$ it follows

$$\mathbf{E}_P^{s^*}(L_k(\hat{P},Q)-L_{k-1}(\hat{P},Q)|\mathcal{F}_{k-1},\mathcal{T})=I(P,Q)-I(P,\hat{P}). \tag{3.7}$$

Denote

$$\xi(k)=L_k(\hat{P},Q)-L_{k-1}(\hat{P},Q)-I(P,Q)+I(P,\hat{P}),\quad \Xi(k)=\sum_{l=1}^{k}\xi(l). \tag{3.8}$$

It is obvious that $\Xi(k)$ is a martingale and under the event $\mathcal{T}$ the conditions of the theorem 8 are satisfied. Therefore, by this theorem

$$\mathbf{E}_P^{s^*}\left(\sup_{Q\in A(\hat{P}),k\le l}|\Xi(k)|\,\middle|\,\mathcal{F}_0,\mathcal{T}\right)\le K_4\sqrt{l}, \tag{3.9}$$

with the same K_4 for all l.

Theorem 6 ([17]) for the processes $\Xi(k)$ with $\varepsilon = 2k_0^{-1} - k(P)I(P,Q) - I(P,\hat{P}))$, $k_0 = \inf_{P\in\mathcal{P}} k(P) > 0$, implies the existence of a positive number b_2 such that

$$\mathbf{P}_P^{s^*}(M=M''|\mathcal{F}_0,\mathcal{T})\le C_2\beta^{b_2}, \tag{3.10}$$

with the same C_2 for all β.

First we estimate the mean length $\mathbf{E}_P^{s^*}N_2$ of the second phase which is the principal part of the total mean length. We get from (3.9) similarly to the proof of the theorem 3 in [22]

$$\mathbf{E}_P^{s^*}(N_2|\mathcal{F}_0,\mathcal{T})\le\frac{|\log\beta|}{k(P)-I(P,\hat{P})}+C_3\sqrt{|\log\beta|}\le$$

$$\le\frac{|\log\beta|}{k(P)-\varepsilon}+C_3\sqrt{|\log\beta|}, \tag{3.11}$$

where $\varepsilon = K_1|\log\beta|^{-b/2}$ and C_3 is independent of β.

Since the second phase is truncated

$$\mathbf{E}_P^{s^*}(N_2|\mathcal{F}_0,\mathcal{F})\le\frac{2|\log\beta|}{k(\hat{P})}\le\frac{2|\log\beta|}{k_0}=L_2. \tag{3.12}$$

The truncation probability of the second phase based on observations of a given loop is bounded from above by $C_1\beta^{b_1}+C_2\beta^{b_2}=p_2$. The first summand of the left-hand side is estimated in (3.6). The second summand's expression follows from (3.10).

Since all attempts to complete the second phase of s^* with the final decision are i.i.d., their amount is distributed geometrically. Hence for the mean length $\mathbf{E}_P^{s^*}N$ we get the following upper bound based on (3.11), (3.12) and the definition of the first phase

$$\mathbf{E}_P^{s^*}N\le\frac{|\log\beta|(k(P))^{-1}+L+L_2p_2+C_4|\log\beta|\varepsilon+C_3\sqrt{|\log\beta|}}{1-p_2}, \tag{3.13}$$

where C_4 is independent of β.

Let us now bound the error probability. It follows from the definition of s^* that

$$\mathbf{P}_P^{s^*}(\mathcal{D}=1-r|\mathcal{F}_0,\mathcal{T})=\mathbf{P}_P^{s^*}\left(L_M(\hat{P},P)>|\log\beta|\,\middle|\,\mathcal{F}_0,\mathcal{T}\right)=$$

$$\mathbf{E}_P^{s^*}\left(\mathbf{E}_{\hat{P}}^{s^*}\left(\exp\left(-L_M(\hat{P},P)\right)\mathcal{I}(L_M(\hat{P},P) > |\log\beta|)\right)\right) \le \beta.$$

Thus:

$$\mathbf{P}_P^{s^*}(\mathcal{D} = 1-r) \le \frac{\beta}{1-p_2}. \tag{3.14}$$

Inequality (3.14) entails that $s^*(\beta, n)$ is α-strategy under β satisfying

$$\frac{\beta}{1-p_2} \le \alpha. \tag{3.15}$$

Since $p_2 \to 0$ as $\alpha \to 0$, we have $|\log\beta| \le \log\alpha| + 1$ under sufficiently small α. Hence (3.13) implies that

$$\mathbf{E}_P^{s^*}N = \frac{|\log\alpha|}{k(P)} + K_2|\log\alpha|^{1-b/2} + K_3\sqrt{|\log\alpha|}$$

for the parameters of $s^*(\beta, n)$ chosen as specified by (3.15).

Proof of Theorem 2 is completed.

We see that the second term in (3.5) is generally of larger order of magnitude than in (2.1). It is remedied by a procedure with more than two phases. The amount of phases depends of b.

Theorem 3. *For every $P \in \mathcal{P}$ under the conditions C1-C3 and C4 with $b \ge \frac{1}{2}$ and appropriate parameters the multi-phased strategy s^* is an α-strategy and*

$$\mathbf{E}_P^{s^*}N \le \frac{|\log\alpha|}{k(P)} + O(\sqrt{|\log\alpha|}).$$

Proof. The three-phased procedure is as follows. The first phase is the same as before. On the second phase we use $N_2 = [|\log\alpha|^{\frac{1+b}{2}}] + 1$ measurements and calculate the statistics

$$L^{(2)}(\hat{P}, Q) = \sum_{i=1}^{N_2} z(\hat{P}, Q, y_i).$$

Let $\hat{\hat{p}}(y)$ be the estimate of p after the second phase. Then we use the statistics

$$L_k^{(3)}(\hat{\hat{P}}, Q) = L^{(2)}(\hat{P}, Q) + \sum_{i=1}^{k} z(\hat{\hat{P}}, Q, y_i)$$

and stop observations as in the two-phased strategy with $L_k(\hat{P}, Q)$ replaced by $L_k^{(3)}(\hat{\hat{P}}, Q)$. The decision and possible call for a new loop are the same as for the two-phased procedure.

The procedure with three phases has the second term of the order $\max\{\frac{1}{2}, \frac{2-b^2-b}{2}\}$. Therefore if $b \ge \frac{\sqrt{5}-1}{2}$ then the procedure with three phases has the second term of the order $\frac{1}{2}$.

Similarly the n-phased procedure can be constructed. We use $n-1$ phases for estimating P. Let $d_i = \frac{1-b^i}{2(1-b)}$. If the i-th phase uses $N_i = [|\log\alpha|^{d_i}] + 1$ measurements then the second term has the order $\max\{\frac{1}{2}, \frac{2-3b+b^n}{2(1-b)}\}$.

For completing the proof we need to replace N_2 with $\sum_{i=2}^{n} N_i$ in (3.11). Thus:

$$\mathbf{E}_P^{s^*}(N_2|\mathcal{F}_0, \mathcal{T}) \le \sum_{i=2}^{n-1}\mathbf{E}_P^{s^*}\left(\frac{N_i}{k(P) - I(P, \hat{P}_{i-1})}|\mathcal{F}_0, \mathcal{T}\right) +$$

$$+\mathbf{E}_P^{s^*}\left(\frac{|\log\beta| - \sum_{i=2}^{n-1}N_i}{k(P) - I(P, \hat{P}_{n-1})}|\mathcal{F}_0, \mathcal{T}\right) + C_3\sqrt{|\log\beta|},$$

where $\hat{P}_i$ is the estimate of P after the i-th phase. Therefore

$$\mathbf{E}_P^{s^*}(N_2|\mathcal{F}_0, \mathcal{T}) \le \frac{|\log\beta|}{k(P)-\varepsilon} + O(\sqrt{|\log\beta|}),$$

where $\varepsilon = K_1|\log\beta|^{-bd_{n-1}}$, . If we take n such that $bd_{n-1} \ge \frac{1}{2}$ then

$$\mathbf{E}_P^{s^*}(N_2|\mathcal{F}_0, \mathcal{T}) \le O(\sqrt{|\log\beta|}). \tag{3.16}$$

If we replace (3.11) with (3.16) then the proof is implied by that of Theorem 2.

4. NON-PARAMETRIC HYPOTHESES TESTING WITH CONTROL

Let the following regularity conditions be satisfied.

C5. Condition C4 is valid with the same values b and $K_1(b)$ for every $u \in U$. Additionally to C4 there exists a sequence of mixed controls $u_n(P), c \ge 0$ and $K_2 = K_2(c)$ such that $u_n(\hat{P}_n)$ is a measurable control for every n and i.i.d.(P) observations $X_1, \dots X_n$ with $P \in \mathcal{P}$, and

$$\mathbf{E}_P|\inf_{Q\in A(P)}(I_{u_n(\hat{P}_n)}(P,Q) - k^*(P))| \le K_2 n^{-c}.$$

Remark 4. In C5 we assume that the estimate $\hat{P}_n$ permits us to approximate $u^*(P)$ in such a way that the preceding bound holds. It can be done e.g. by approximating a quasi-homogeneous smooth $\mathcal{P}$ as in Remark 3 by finite-dimensional exponential families using the methodology of [21].

Our procedure $S^* = S^*(\beta, n)$, as before, has conditionally i.i.d. loops until the final success almost similar to (3.3). If an analogue of (3.4) holds, a new loop begins. Every loop contains two phases.

For every $u \in U$ based on $L = [\sqrt{|\ln\alpha|}] + 1$ independent observations with this control we estimate measure P^u in the first phase as in the previous case. Let $\hat{P}^u$ be the estimate for P^u as before.

We use the control $u(i) = u_L(\hat{P})$ for the i-th measurement of the second phase and stop observations and take the decision δ as in the strategy s^*.

Theorem 4. *Under the conditions C1-C5 for every $P \in \mathcal{P}$ under appropriate parameters S^* is an α-strategy and*

$$\mathbf{E}_P^{S^*}N \le \frac{|\log\alpha|}{k^*(P)} + K_3|\log\alpha|^{1-d/2} + K_4 3\sqrt{|\log\alpha|}$$

where $d = \min(b, c)$, the numbers K_3 and K_4 do not depend on α.

Proof is similar to that of Theorem 2. Since we use the control $u_L(\hat{P})$ instead of $u^*(P)$, (3.11) takes the form

$$\mathbf{E}_P^{S^*}(N_2|\mathcal{F}_0, \mathcal{T}) \le \frac{|\log\beta|}{k^*(P)-\varepsilon} + O(\sqrt{|\log\beta|}),$$

where $\varepsilon = K_1|\log\beta|^{-\frac{b}{2}} + K_2|\log\beta|^{-\frac{c}{2}}$.

A further strengthening of the upper bound in theorem 4 can be obtained by studying multi-phased strategies as in section 2.

5. EXTENSIONS AND APPLICATIONS

5.1. General Risk

Our theorems 1 and 2 can be extended to a general loss function with power growth of the strategy length. Let $g : \mathbf{R}^+ \to \mathbf{R}^+$ be continuous, increasing to infinity and

i. $g(an) \le K_g a^k g(n)$ for all $a > 1$, all large n, and some positive K_g and k;

ii. $1 - c_g \frac{k}{n} \le \frac{g(n+k)}{g(n)} \le 1 + C_g \frac{k}{n}$, with the same constants c_g and C_g for all n and k.

Let $\mathbf{E}_P^s g(N)$ be the risk of the strategy s.

Theorem 5. *Under the conditions C1 any α-admissible strategy s satisfies*

$$\mathbf{E}_P^s g(N) \ge g\left(\frac{|\log\alpha|}{k(P)}\right)\left(1 + O(\frac{1}{\sqrt{|\log\alpha|})}\right).$$

Under the conditions C1-C4 and appropriate parameters, s^* is an α-strategy and

$$\mathbf{E}_P^{s^*} g(N) \le g\left(\frac{|\log\alpha|}{k(P)}\right)\left(1 + K_2|\log\alpha|^{-b/2} + K_3|\log\alpha|^{-\frac{1}{2}}\right),$$

with the same constants K_2 and K_3 for all α.

Proof is a rather straightforward generalization of that for Theorem 1 and 2 along the lines of [17].

5.2. Change-Point Detection

A version of our procedure $s^* = s^*(\beta, n)$ is also applicable for the non-parametric detection of abrupt change in the distribution of i.i.d. sequences as outlined in section 1.3.

Denote $\mathbf{X}_t = (x_t, x_{t+1}, \dots)$, where x_i is the i-th measurement. For every $t \ge 1$ based on the sequence of measurements $\mathbf{X}_t$ we construct one loop of the procedure $s^* = s^*(\beta, n)$ which is denoted by $s_t^*(\beta, n)$ for estimating N_t. The loop contains two phases.

Based on the first $L = [\sqrt{|\ln\beta|}] + 1$ observations of the loop we construct an estimate $\hat{P}$ of the density function P as outlined in section 2.3.

If $\hat{P} \in \mathcal{P}_0$ then we stop the loop and take $N_t = \infty$. If $\hat{P} \in \mathcal{P}_1$ then we perform the following test in the second phase. Let us numerate measurements of the second phase anew and introduce $L_k(\hat{P}, Q) = \sum_{i=1}^k z(\hat{P}, Q, y_i)$.

We stop observations at the first moment M such that

$$\inf_{Q \in \mathcal{P}_0} L_M(\hat{P}, Q) > -\log\beta \tag{5.1}$$

or

$$M > 2k(\hat{P})^{-1}|\log\beta| \tag{5.2}$$

and take $N_t = t + M$ if (5.1) holds and $1 - r$ is the index of the set $A(\hat{P})$. If (5.2) holds then we take $N_t = \infty$.

Theorem 6. *Under the condition C1 for sufficiently small α we have the following lower bound* $\bar{\mathbf{E}}^s(N) \ge |\log\alpha| I_1^{-1} + O(\log|\log\alpha|)$.

Under the conditions C1-C4 and appropriate parameters of s^* the procedure for the change-point detection's problem is an α-strategy and

$$\bar{\mathbf{E}}^s(N) \le |\log\alpha| I_1^{-1} + K_5|\log\alpha|^{1-b/2} + K_6\sqrt{|\log\alpha|}, \tag{5.3}$$

with the same constants K_5 and K_6 for all α.

Proof. The lower bound follows from the proof in [18] if we put $\varepsilon = |\log\alpha|^{-1}$.

The condition (1.9) follows from the theorem 2 in ([18] and (5.3) follows from our Theorem 2.

For a general loss function from section 5.1 and $\mathbf{E}_P^{s^*} g((N - \nu + 1)^+)$ the result of Theorem 4 implies an upper bound for a general risk of the change-point detection.

5.3. Sequential testing homogeneity

Let independent observations may be taken from $m \geq 2$ populations with distributions $P_1, \dots, P_m$ on $(X, \mathcal{B}, \mu)$, $P_i \in \mathcal{P}$, and the conditions C1-C4 valid for $\mathcal{P}$. We test $H_0 : P_1 = \dots = P_m$ versus $H_1 : \max_{i,j} d(P_i, P_j) \geq \Delta > 0$, where d is I-uniformly continuous distance on $\mathcal{P}$. Applying control u means that we take an observation from the u-th population. This is a particular case of controlled experiments for testing the hypothesis $H_0 : P \in \mathcal{P}_0$ versus $H_1 : P \in \mathcal{P}_1$ with the indifference zone $P \in \mathcal{P}_+$ where $P = (P_1, \dots, P_m), P_i \in \mathcal{P}$, $\mathcal{P}_0 = \{P : P_1 = \dots, P_m\}$, $\mathcal{P}_1 = \{P : \max_{i,j} d(P_i, P_j) \geq \Delta > 0\}$, $\mathcal{P}_+ = \{P : 0 < \max_{i,j} d(P_i, P_j) < \Delta\}$.

If $P \in \mathcal{P}_0$ then

$$k^*(P) = \frac{1}{m} \inf_{Q,R:d(Q,R)\geq\Delta} (I(P,Q) + I(P,R)) \tag{5.4}$$

and $u^*(P) = (\frac{1}{m}, \dots, \frac{1}{m})$. If $P \in \mathcal{P}_1$ then

$$k^*(P) = \max_{u \in U^*} f(u), \tag{5.5}$$

where

$$f(u) = \sum_{i=1}^{m} \kappa_i I(P_i, P^u) + \inf_{Q \in \mathcal{P}} I(P^u, Q),$$

$p^u = \sum_{i=1}^m \kappa_i p_i, u = (\kappa_1, \dots, \kappa_m) \in U^*$, and

$$u^*(P) = \arg\max_{u \in U^*} f(u). \tag{5.6}$$

Control $u^*(P)$ exists since $f(u)$ is a continuous function of $\kappa_1, \dots, \kappa_m$.

For $P \in \mathcal{P}_+$ we calculate $k_1(P)$ by (5.5) and

$$k_0(P) = \frac{\Pi_{i=1}^m a_i}{\sum_{i=1}^m \Pi_{j=1, j\neq i}^m a_i},$$

where

$$a_i = \inf_{Q,R:d(Q,R)\geq\Delta} (I(P_i,Q) + I(P_i,R)), \tag{5.7}$$

and $k^*(P) = \min(k_0(P), k_1(P))$. If $k^*(P) = k_0(P)$ then $u^*(P) = (\kappa_1^0, \dots, \kappa_m^0)$, where

$$\kappa_i^0 = \frac{\Pi_{j\neq i} a_i}{\sum_{k=1}^m \Pi_{j\neq k}^m a_i}.$$

If $k^*(P) = k_1(P)$ then $u^*(P)$ is given by (5.6).

If we verify the condition C5 then we can use the general strategy from the previous section.

Theorem 7. *Let $X = [0,1]$, $\mathcal{P}$ be such that $\log p \in W_2^r, ||D^r \log p||_2 \leq K$, $r > 1$, $\log p$ be a periodic function, and μ be Lebesgue measure. Under the conditions C1-C3 and appropriate parameters S^* is an α-strategy and*

$$\mathbf{E}_P^{S^*} N \leq \frac{|\log\alpha|}{k^*(P)} + K_7 |\log\alpha|^{-d} + K_8 \sqrt{|\log\alpha|} \tag{5.8}$$

where $d = \frac{r-1}{4r}$ and the constants K_7 and K_8 are the same for all α.

Proof. The regularity conditions on $\log p$ force the density to be strictly positive and finite on $[0,1]$. If we approximate $f = \log p$ by

$$f_l = \beta_0 + \sum_{k=1}^{l} \beta_{2k}\sqrt{2}\cos(2\pi kx) + \sum_{k=1}^{l} \beta_{2k+1}\sqrt{2}\sin(2\pi kx) \tag{5.9}$$

then

$$||f - f_l||_\infty \leq K c_r l^{-(r-1/2)}, \tag{5.10}$$

where $c_r = \sqrt{2r-1}\pi^{-r}$ ([20], p. 1365).

Let $\mathcal{A}_l$ be the set of measures with $\log p$ given by (5.9) and $Q^* = Q_l^*$ be the information projection ([23]) of $Q \in \mathcal{P}$ into $\mathcal{A}_l$. If we replace in (5.4), (5.5), (5.7) $\mathcal{P}$ with $\mathcal{A}_l$, then it follows from (5.10) that

$$k^*(P) - I_{u_l(P^*)}(P, A(P) = O(l^{-(r-1/2)}).$$

If $\hat{P}^*$ is the ML-estimate of coefficients $\beta_k, k = 1, \dots 2l+1$ in $\mathcal{A}_l$, then smoothness of densities in $\mathcal{A}_l$ implies

$$\mathbf{E}_P(k^*(P) - (I_{u_l(\hat{P}^*_L)}(P, A(P)) \leq O(l^{-(r-1/2)}) + O(\sqrt{\frac{l}{L}}),$$

where L is the coinciding amount of observations from every population at the first phase. Optimizing in l we verify the condition C5 with $c = \frac{r-1}{2r}$. Therefore Theorem 4 implies (5.8) .

6. APPENDIX

Consider a family of martingale-differences $(z(\varphi, x_n), \mathcal{F}_n)$ with respect to a flow $\mathcal{F}_n$ of σ-algebras, i.e.

$$\mathbf{E}\left(z(\varphi, x_n)|\mathcal{F}_{n-1}\right) = 0$$

for all $n = 1, 2, \dots$ under all φ from an arbitrary set Φ. Here we bound from above $\sup_{\varphi\in\Phi} |z_n(\varphi)|$ and $\sup_{\varphi\in\Phi} |z_\tau(\varphi)|$ where $z_n(\varphi) = \sum_{k=1}^n z(\varphi, x_k)$ is a martingale and τ is a stopping time with respect to $\mathcal{F}_n$. We assume the following regularity conditions.

B1.

$$\mathbf{E}\left((z(\varphi, x_n))^2|\mathcal{F}_{n-1}\right) < c.$$

uniformly over $\varphi \in \Phi$ and $n = 1, 2, \dots$

B2. The function $z(\varphi, x)$ is differentiable w.r.t. x, $X = R$, and

$$D = \int_{-\infty}^{+\infty} z_1(x)\, (p(x)q(x))^{1/2}\, dx < \infty, \tag{6.1}$$

where

$$z_1(x) = \sup_{\varphi\in\Phi} |\frac{\partial z(\varphi, x)}{\partial x}|,$$

and the functions $p(x)$ and $q(x)$ are such that the following inequalities

$$\mathbf{P}(x_n \leq x|\mathcal{F}_{n-1}) \leq p(x), \quad \mathbf{P}(x_n > x|\mathcal{F}_{n-1}) < p(x)$$

are valid for all $x \in X$ and $n = 1, 2, \dots$

Theorem 8. *If B1 and B2 are satisfied, then the following inequality is valid for any stopping time τ with bounded mean*

$$\mathbf{E}\left(\max_{m\leq\tau} \sup_{\varphi\in\Phi} |z_m(\varphi)|\right) \leq BD\sqrt{\mathbf{E}(\tau)}, \tag{6.2}$$

where D is given by (6.1).

Proof. The following notation is used:

$$F_m(x) = \sum_{k=1}^m I(x_k \leq x), \tilde{F}_m(x) = \sum_{k=1}^m \mathbf{E}\left(I(x_k \leq x)|\mathcal{F}_{k-1}\right), G_m(x) = F_m(x) - \tilde{F}_m(x).$$

Since it is clear that almost surely

$$\lim_{x\to-\infty}\tilde{F}_m(x)=0,\ \lim_{x\to-\infty}F_m(x)=0,\ \lim_{x\to+\infty}\tilde{F}_m(x)=m,\ \lim_{x\to+\infty}G_m(x)=0,$$

we get the following chain of equalities

$$z_m(\varphi)=\int_R z(\varphi,x)dF_m(x)=\int_R z(\varphi,x)d\tilde{F}_m(x)+\int_R z(\varphi,x)dG_m(x)=$$

$$\int_R z(\varphi,x)dG_m(x)=z(\varphi,x)G_m(x)\Big|_{-\infty}^{\infty}-\int_R\frac{\partial z(\varphi,x)}{\partial x}G_m(x)dx=-\int_R\frac{\partial z(\varphi,x)}{\partial x}G_m(x)dx.$$

The property

$$\int_R z(\varphi,x)d\tilde{F}_m(x)=0$$

is used here implied by the fact that the process $z_m(\varphi)$ is a martingale. Hence

$$\sup_{\varphi\in\Phi}|z_m(\varphi)|\le\int_R\sup_{\varphi\in\Phi}\left|\frac{\partial z(\varphi,x)}{\partial x}\right||G_m(x)|dx. \tag{6.3}$$

Since obviously $G_m(x)$ is a martingale w.r.t. the flow $\mathcal{F}_m$, the Davis inequality ([24]) implies that

$$\mathbf{E}\left(\max_{m\le\tau}|G_m(x)|\right)\le B\left(p(x)q(x)\right)^{1/2}\sqrt{\mathbf{E}(\tau)}, \tag{6.4}$$

where B is a universal constant.

Inequalities (6.3) and (6.4) entail the statement (6.2).

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